A TREATISE

ON

PRACTICAL GEOMETRY,

MENSURATION,

Conic Sections, Gauging, and Land-Surveying,

WITH AN ESSAY

ON

THE SPECIFIC GRAVITY OF BODIES, THE TONNAGE OF SHIPS, THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS, AND ARITHMETIC OF INFINITIES,

FOR THE USE OF THE

IRISH NATIONAL SCHOOLS.

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A TREATISE

ON

MENSURATION

IN THEORY AND PRACTICE.

SECTION I.

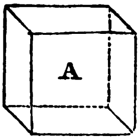
PRACTICAL GEOMETRY.

DEFINITIONS.

1. GEOMETRY teaches and demonstrates the properties of all kinds of magnitudes, or extension; as solids, surfaces, lines, and angles.

2. Geometry is divided into two parts, theoretical and practical. Theoretical Geometry treats of the various properties of extension abstractedly; and Practical Geometry applies these theoretical properties to the various purposes of life. When length and breadth only are considered, the science which treats of them is called Plane Geometry; but when length, breadth and thickness, are considered, the science which treats of them is called Solid Geometry.

3. A *Solid* is a figure, or body, having three dimensions, viz. length, breadth, and thickness, as A.



The boundaries of a solid are surfaces, or superficies.

4. A *Superficies*, or surface, has length and breadth only; as B.

The boundaries of a superficies are lines.

5. A *Line* is length without breadth, and is formed by the motion of a point; as BC.

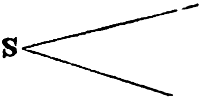
The extremities of a line are points.

Note.—It is likewise necessary to conceive that a line is composed of an infinite number of points, each less than any assignable quantity.

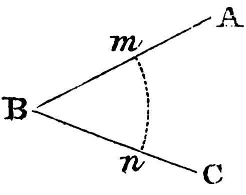
6. A *Straight Line* is the shortest distance between two points, and lies evenly between these two points.

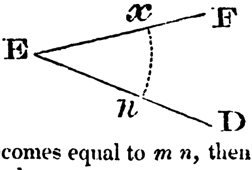
7. A *Point* is that which has no parts or magnitude; it is indivisible; it has not length, breadth, or thickness. If it had length, it would then be a line; were it possessed of length and breadth, it would be a superficies; and had it length, breadth, and thickness, it would be a solid. Hence a point is void of length, breadth, and thickness, and is only the creature of imagination.

8. A *Plane rectilineal Angle* is the inclination of two right lines which meet in a point, but are not in the same direction; as S.



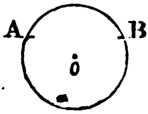
9. One angle is said to be less than another, when the lines which form the angle are nearer to each other; measuring at equal distances from the points in which the lines meet. Take Bn, Bm, Ex, and En, equal to one another; then if $m\,n$ be greater than $x\,n$, the angle ABC is greater than the angle FED. By conceiving the point A to move towards C, till $m\,n$ becomes equal to $x\,n$, the angles at B and E would then be equal; or by conceiving the point F to recede from D, till $x\,n$ becomes equal to $m\,n$, then the angles at B and E would be equal.



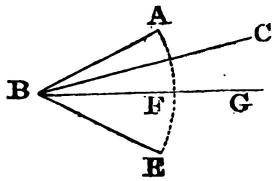


Hence it appears that the nearer the extremities of the lines forming an angle approach each other, while the point at which they meet remains fixed, the less the angle; and the farther the extreme points recede from each other, the vertical point remaining fixed, as before, the greater the angle.

10. A *Circle* is a plane figure contained by one line called the circumference, which is every where equally distant from a point within it, called its centre; as o: and an arc of a circle is any part of its circumference; as A B.



11. The magnitude of an angle does not consist in the length of the lines which form it: the angle CBG is less than the angle ABE, though the lines CB, GB are longer than AB, EB.

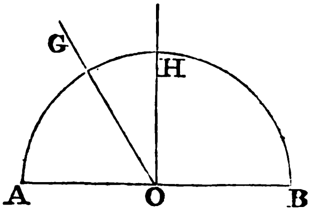


12. When an angle is expressed by three letters, as ABE, the middle letter always stands at the angular point, and the other two any where along the sides; thus the angle A B E is formed by A B and B E. The angle A B G by A B and G B.

13. In equal circles, angles have the same ratio to each other as the arcs on which they stand, (33. VI). Hence also, in the same, or equal circles, the angles vary as the arcs on which they stand; and therefore the arcs may be assumed as proper measures of angles. Every angle then is measured by an arc of a circle, described about the angular point as a centre; thus the angle ABE is measured by the arc A E; the angle A B G by the arc A F.

14. The circumference of every circle is generally divided into 360 equal parts, called degrees; and every degree into 60 equal parts, called minutes; and each minute into 60 equal parts, called seconds. The angles are measured by the number of degrees contained in the arcs which subtend them; thus, if the arc A E contain 40 degrees, or the ninth part of the circumference, the angle A B E is said to measure 40 degrees.

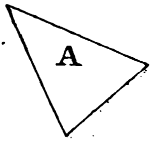
15. When a straight line HO, standing on another AB, makes the angle HOA eqnal to the angle HOB; each of these angles is called a right angle; and the line HO is said to be a perpendicular to AB. The measure of the angle HOA is 90 degrees, or the fourth part of 360 degrees. Hence a right angle is 90 degrees.



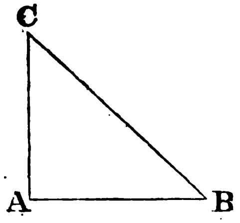
16. An acute angle is less than a right angle; as AOG, or GOH.

17. An obtuse angle is greater than a right angle; as GOB.

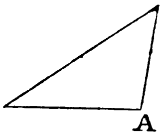
18. A *plane Triangle* is the space enclosed by three straight lines, and has three angles; as A.



19. A *right angled Triangle* is that which has one of its angles right; as ABC. The side BC, opposite the right angle is called the hypothenuse; the side AC is called the perpendicular; and the side AB is called the base.

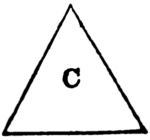


20. An *obtuse angled Triangle* has one of its angles obtuse; as A.

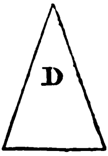


21. An *acute angled Triangle* has all its three angles *acute, as in the figure* A.

22. An *equilateral Triangle* has its three sides equal, and also its three angles; as C.



23. An *isosceles Triangle* is that which has two of its sides equal, and the third side either greater or less than either of the equal sides; as D.

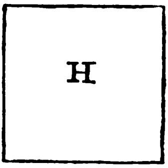


24. A *scalene Triangle* is that which has all its sides unequal; as E.

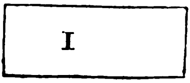
25. A *quadrilateral* figure is a space included by four straight lines. If its four angles be right, it is called a rectangular parallelogram.

26. A *Parallelogram* is a plane figure bounded by four straight lines, the opposite ones being parallel; that is, if produced ever so far, would never meet.

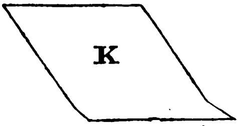
27. A *Square* is a four-sided figure, having all its sides equal, and all its angles right angles; as H.



28. An *Oblong*, or rectangle, is a right angled parallelogram, whose length exceeds its breadth; as I.

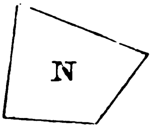


29. A *Rhombus* is a parallelogram having all its sides equal, but its angles not right angles; as K.



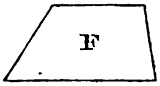
30. A *Rhomboid* is a parallelogram having its opposite sides equal, but its angles are not right angles, and its length exceeds its breadth; as M.

31. A *Trapezium* is a figure included by four straight lines, no two of which are parallel to each other; as N.



A line connecting any two of its angles is called a diagonal.

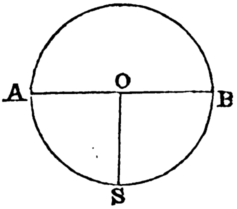
32. A *Trapezoid* is a four-sided figure having two of its opposite sides parallel, but the remaining two not parallel; as F.



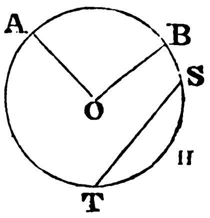
33. Multilateral Figures, or Polygons, are those which have more than four sides. They receive particular names from the number of their sides. Thus, a *Pentagon* has five sides; a *Hexagon*, has six sides; a *Heptagon*, seven; an *Octagon*, eight; a *Nonagon*, nine; a *Decagon*, ten; an *Undecagon*, eleven; and a *Duodecagon*, has twelve sides.

If all the sides of each figure be equal, it is called a regular polygon; but if unequal, an irregular polygon.

34. The *Diameter* of a circle is a straight line passing through the centre, and terminated by the circumference; thus A B is the diameter of the circle. The diameter divides the circle into equal parts, each of which is called a semi-circle; the diameter also divides the circumference into two equal parts, each containing 180 degrees. Any line drawn from the centre to the circumference is called the radius, as A O, O B, or O S. If O S be drawn from the centre perpendicular to A B, it divides the semicircle into two equal parts, A O S and B O S, each of which is called a quadrant, or one-fourth of the circle; and the arcs A S and B S contain each 90 degrees, and they are said to be the measure of the angles A O S and B O S.



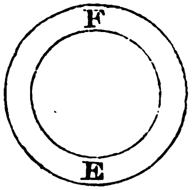
35. A *Sector* of a circle is that part of the circle comprehended under two *Radii*, not forming one line, and the part of the circumference between them. From this definition it appears that a sector may be either greater or less than a semi-circle; thus A O B is a sector, and is less than a semi-circle; and the remaining part of the circle is a sector also, but is greater than a semi-circle.



36. A *Chord* of an arc is a straight line joining its extremities, and is less than the diameter; T S is the chord of the arc T H S, or of the arc T A B S.

37. A *Segment* of a circle is that part of the circle contained between the chord and the circumference, and may be either greater or less than a semi-circle; thus T S H T and TABST are segments, the latter being greater than a semicircle, and the former less.

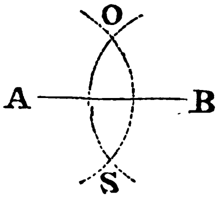
38. *Concentric circles* are those having the same centre, and the space included between their circumferences is called a ring; as F E.



PROBLEM I.

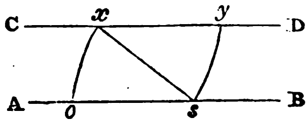
To bisect a given straight line A B*; that is, to divide it into two equal parts.*

From the centres A and B, with any radius, greater than half the given line A B, describe two arcs intersecting each other at O and S, then the line joining O S will bisect A B.



PROBLEM II.

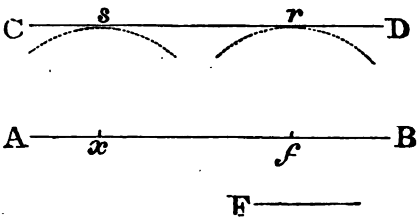
Through a given point x to draw a straight line C D *parallel to a given straight line* A B.



In A B take any point *s*, and with the centre *s* and radius *s x* describe the arc *o x*; with *x* as a centre and the same radius *s x*, describe the arc *s y*. Lay the extent *o x* taken on the compasses from *s* to *y*; through *x y* draw C D, which will be parallel to A B.

PROBLEM III.

To draw a straight line C D *parallel to* A B, *and at a given distance* F, *from it.*

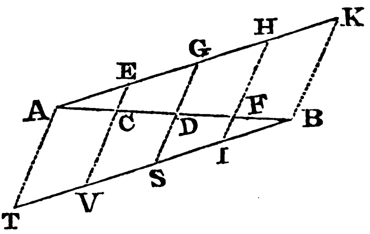


In A B take any two points *x*, *f*; and from the two points as centres, with the extent F, taken on the compasses, describe two arcs *s r*; then draw a line C D touching these arcs at *r* and *s*, and it will be at the given distance from A B, and parallel to it.

PROBLEM IV.

To divide a straight line A B *into any number of equal parts.*

Draw A K making any angle with A B; and through B draw B T parallel to A K; take any part A E and repeat it as often as there are parts to be in A B, and from the point B on the line B T, take B I, I S, S V, and V T equal to the parts taken on the line A K; then join A T, E V, G S, *H I, and* K B, which *will divide the line* A B *into the number of* equal parts required, as AC, CD, DF, FB.

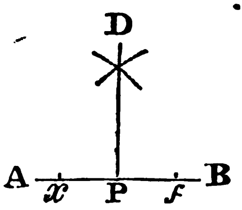


PROBLEM V.

From a given point P *in a straight line* A B *to erect a perpendicular.*

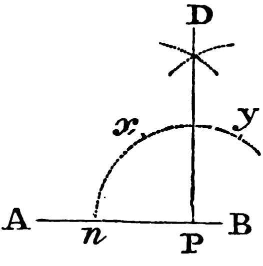
1. *When the given point is in, or near the middle of the line.*

On each side of the point P take equal portions, P x, P f; and from the centres x, f, with any radius greater than P x, describe two arcs, cutting each other at D; then the line joining D P will be perpendicular to A B.



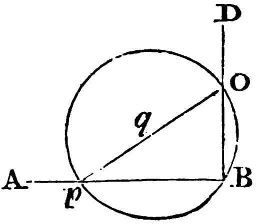
Or thus:

From the centre P, with any radius P n describe an arc $n\,x\,y$; set off the distance P n from n to x, and from x to y; then from the points x and y with the same, or any other radius, describe two arcs intersecting each other at D; then the line joining the points D and P will be perpendicular to A B.



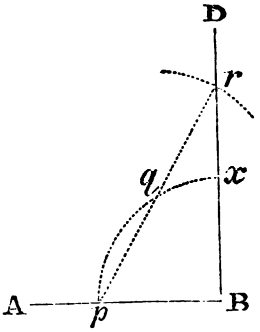
2. *When the point* P *is at the end of the line.*

From any centre q out of the line, and with the distance q B as radius, describe a circle, cutting A B in p; draw $p\,q$ O; and the line joining the points O, B will be perpendicular to A B.



Or thus:

Set one leg of the compasses on B, and with any extent B p describe an arc P x; set off the same extent from p to q; join $p\,q$; from q as a centre, with the extent $p\,q$ as radius, describe an arc r; produce $p\,q$ to r, and the line joining r B will be perpendicular to A B.

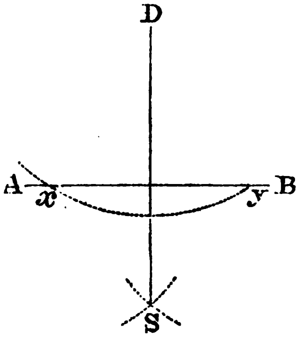


PROBLEM VI.

From a given point D *to let fall a perpendicular upon a given line* A B.

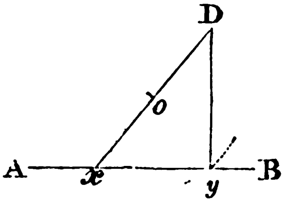
1. *When the point is nearly opposite the middle of the given line.*

From the centre D, with any radius, describe an arc $x\,y$, cutting A B in x and y; from x and y as centres, and with the same distance as radius, describe two arcs cutting each other at S; then the line joining D and S will be perpendicular to A B.



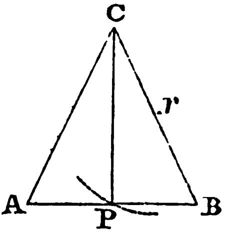
2. *When the point is nearly opposite the end of the given line, and when the given line cannot be conveniently produced.*

Draw any line D x, which bisect in o; from o as a centre with the radius $o\,x$ describe an arc cutting A B in y; then the line joining D y will be perpendicular to A B.

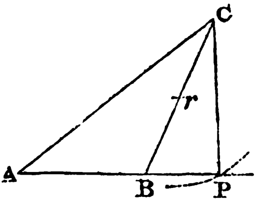


PROBLEM VII.

To draw a perpendicular, from any angle of a triangle A B C, *to its opposite side.*



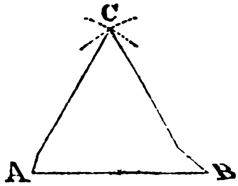
Bisect either of the sides containing the angle from which the perpendicular is to be drawn, as B C in the point r; then with the radius r C, and from the centre r, describe an arc, cutting A B, (or A B produced if necessary, as in the second figure,) in the point P; the line joining C P will be perpendicular to A B, or to A B produced.



PROBLEM VIII.

Upon a given right line A B *to describe an equilateral triangle.*

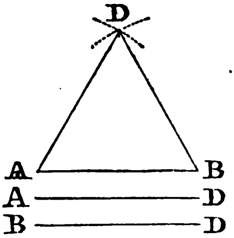
From the centres A and B, with the given line A B as radius, describe two arcs cutting each other at C; then the lines drawn from the point C to the points A and B will form, with the given line A B, an equilateral triangle, as A B C.



PROBLEM IX.

To make a triangle whose sides shall be equal to three given right lines A B, A D, *and* B D.

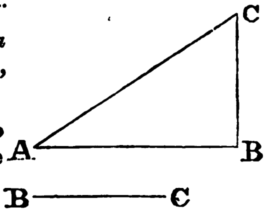
From the centre A with the extent A D, on the compasses, describe an arc, and from the centre B with the radius B D describe another arc cutting the former at D; then join D A, D B, and the sides of the triangle A B D will be respectively equal to the three given right lines.



PROBLEM X.

Two sides A B, *and* B C *of a right angled triangle being given, to find the hypothenuse.*

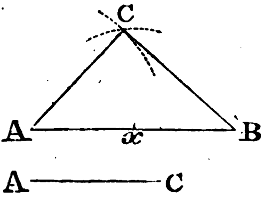
Place B C at right angles to A B; join A C and it will be the hypothenuse required.



PROBLEM XI.

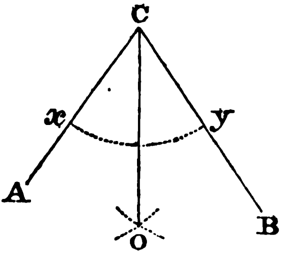
The hypothenuse A B, *and one side* A C, *of a right angled triangle being given, to find the other side.*

Bisect A B in x; with the centre x, and x A as radius, describe an arc; and with A as a centre, and A C as radius, describe another arc cutting the former at C; then join A C and C B; and A B C will be a right angled triangle, and B C the required side.



PROBLEM XII.

To bisect a given angle; that is, to divide it into two equal parts.

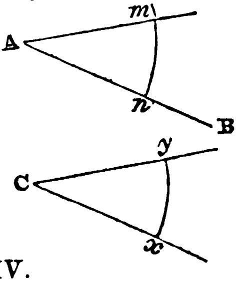


Let A C B be the angle to be bisected.

From C as a centre with any radius C x, describe the arc $x\,y$; from the points x and y as centres with the same radius, describe two arcs cutting each other at O; join O C, and it will bisect the angle at A C B.

PROBLEM XIII.

At a given point A *in a given right line* A B, *to make an angle equal to the given angle* C.



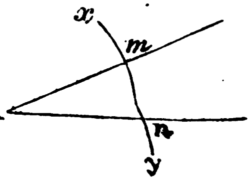
From the centre C with any radius C y describe an arc $x\,y$; and from the centre A with the same radius describe another arc, on which take the distance $m\,n$ equal to $x\,y$; then a line drawn from A through m will make the angle m A n equal to the angle x C y.

PROBLEM XIV.

To make an angle containing any proposed number of degrees.

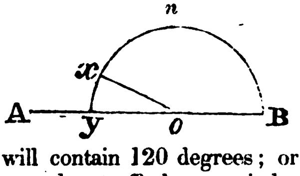
1. *When the required angle is less than a quadrant, as* 40 *degrees.*

Take on the compasses the extent of 60 degrees from the line of chords, marked cho. on the scale; and with this chord of 60 degrees as radius, and the centre A, describe an arc $x\,y$; take from the line of chords 40 degrees, which set off from n to m; from A draw a line through m; *and the angle* m A n will contain *40 degrees.*



2. *When the required angle is greater than a quadrant, as 120 degrees.*

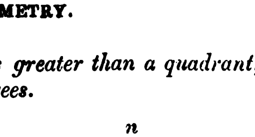
From the centre *o*, with the chord of 60 degrees, as radius, describe the semi-circle *y x n* B; set off the chord of 90 degrees from B to *n*, and the remaining 30 degrees from *n* to *x*; join *o x*; and the angle B *o x* will contain 120 degrees; or subtract 120 from 180 degrees, and set off the remainder (60 degrees) taken from the line of chords from *y* to *x*; then join *x o*, and B *o x* will contain 120 degrees, as before.



PROBLEM XV.

An angle being given, to find, by a scale of chords, how many degrees it contains.

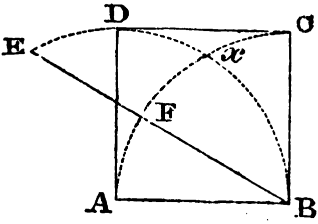
From the vertex A as centre with the cord of 60 degrees as radius describe an arc *x y*; take the extent *x y* on the compasses, and setting one foot at the beginning of the line of chords, the other leg will reach to the number of degrees which the angle contains: but if the extent on the compasses should reach beyond the scale, find the number of degrees in *x y*, which deducted from 180, will leave the degrees in the angle B *o x*. In this, and in the second case of the last Problem, B *o* is to be produced to A.



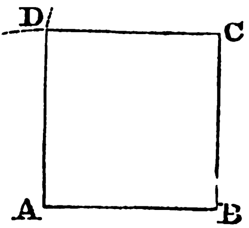
PROBLEM XVI.

Upon a given right line A B, *to construct a square.*

With the distance A B as radius, and A as a centre, describe the arc E D B; and with the distance A B as radius, and B as a centre, describe the arc A F C, cutting the former in *x*; make B *x* equal to *x* E; join E B; make *x* C and *x* D each *equal to* A F, or F *x*; then join the points A D C B, and *they will form a* square.



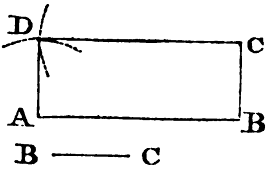
Or thus:



Draw B C at right angles to A B, and equal to it; then from the centres A and C, with the radius A B and C B, describe two arcs cutting each other at D; join D A and D C, which will complete the square.

PROBLEM XVII.

To make a rectangular parallelogram of a given length and breadth.

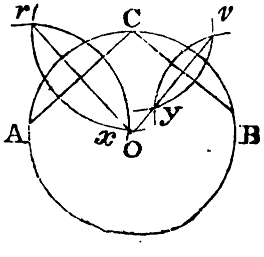


Let A B be the length, and B C the breadth.

Erect B C at right angles to A B; through C and A draw C D and A D, parallel to A B and B C.

PROBLEM XVIII.

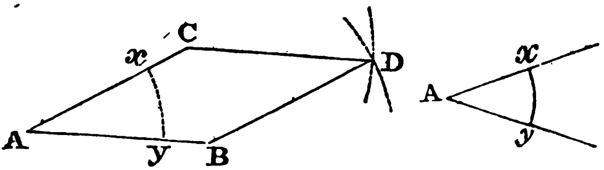
To find the centre of a given circle.



Draw any two cords A C, C B; from the points A, C, B, as centres with any radius greater than half the lines, describe four arcs cutting in *r x*, and *y v*, draw *r x* and *y v*, and produce them till they meet in O, the centre.

PROBLEM XIX.

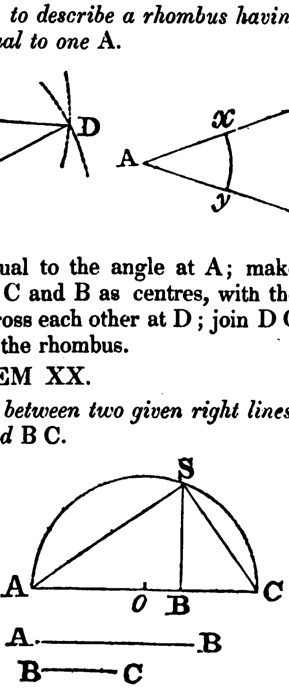
Upon a given right line A B, *to describe a rhombus having an angle equal to one* A.



Make the angle C A B equal to the angle at A; make A C equal to A B; then from C and B as centres, with the radius A B describe two arcs cross each other at D; join D C and D B, which will complete the rhombus.

PROBLEM XX.

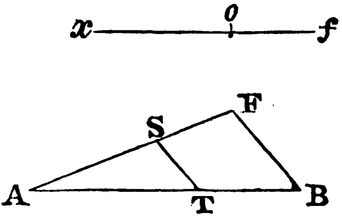
To find a mean proportional between two given right lines, A B *and* B C.



Place A B and B C in one straight line; bisect A C in *o*; from *o* as a centre, with A *o*, or *o* C as radius, describe a semi-circle A S C; erect the perpendicular B S, and it will be a mean proportional between A B and B C; that is, A B : B S :: B S : B C.

PROBLEM XXI.

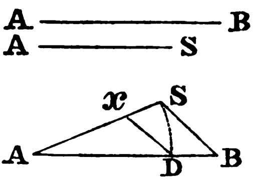
To divide a given right line A B *into two such parts, as shall be to each other, as* x o *to* o f.



From the point A draw A S equal to *x o*, and produce it till F S becomes equal to *o f*; join F B; and draw S T parallel to F B; then will A T : *T B :: x o : o f*.

PROBLEM XXII.

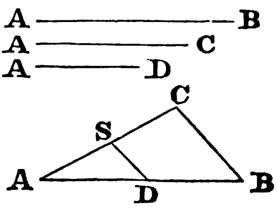
To find a third proportional to two given right lines A B, A S



Place A B and A S so as to make any angle at A; from the centre A, with the distance A S describe the arc S D; then draw D *x* parallel to B S, and A *x* will be the third proportional required; that is, A B : A S : : A S : A *x*.

PROBLEM XXIII.

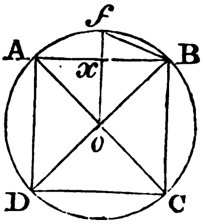
To find a fourth proportion to three given right lines, A B, A C, *and* A D.



Place the right lines A B and A C so as to make any angle at A; on A B set off A D; join B C; and draw D S parallel to it; then A S will be the fourth proportional required, viz. A B : A C : : A D : A S.

PROBLEM XXIV.

In a given circle to describe a square.

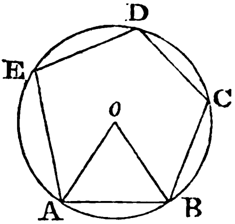


Draw any two diameters A C, D B at right angles to each other; then join their extremities, and the figure A B C D will be a square inscribed in the given circle.

If a line be drawn from the centre *o* to the middle of A B, and produced to *f*; the line joining *f* B will be the side of an octagon inscribed in the circle.

PROBLEM XXVI.

To make a regular polygon on a given right line A B.



Divide 360 degrees by the number of sides contained in the polygon; deduct the quotient from 180 degrees, and the remainder will be the number of degrees in each angle of the polygon. At the points A and B make the angles *o* A B and *o* B A each equal to half the angle of the polygon; then from *o* as a centre, and with *o* A or *o* B as radius, describe a circle, in which place A B continually*.

Or thus:

Take the given line A B from the scale of equal parts, and multiply the number of equal parts in it by the number in the third column of the following table, answering to the given number of sides; the product will give the number of equal parts in the radius A *o*, or *o* B, which taken from the scale of equal parts on the compasses, will give the radius, with which describe a circle, and place in it the line A B continually, as shown in the first method.

---

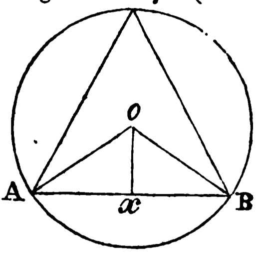
* *Demonstration.* 360 degrees divided by the number of sides, will give the arc A B, which is the measure of the angle A *o* B, and A *o* B being deducted from 180 degrees, will leave the sum of the equal angles *o* A B, *o* B A; therefore half the remainder will give the angle *o* A B or *o* B A, which is half of the angle E A B or A B C; hence the angle A *o* B taken from 180, will leave the angle E A B or A B C.

TABLE I.

When the side of the Polygon is 1.

No. of Sides.	Name of the Polygon.	Radius of the Circumscribing Circle.	Angle O A B, or O B A.
3	Trigon	·5773503	30
4	Tetragon	·7071068	45
5	Pentagon	·8506508	54
6	Hexagon	1, *Side=Radius.*	60
7	Heptagon	1·1523825	$64\frac{2}{7}$
8	Octagon	1·3065630	$67\frac{1}{2}$
9	Nonagon	1·4619022	70
10	Decagon	1·6186340	72
11	Undecagon	1·7747329	$73\frac{7}{11}$
12	Duodecagon	1·9318516	75

Demonstration. 360 degrees being divided by 3 (for the trigon,) will evidently give the angle A *o* B equal to 120 degrees, which being deducted from 180 degrees, will give the sum of the equal angles *o* A B and *o* B A equal to 60 degrees, half of which, viz. 30 degrees, is equal to the angle *o* A B, or *o* B A. In a similar way the angle made by the side of any of the other regular figures in the Table with the radius of the circumscribing circle may be found. Having the angle *o* A B, or *o* B A, and the side A B, which in each of the above figures is 1; we can easily discover the side *o* B, the radius of the circumscribing circle; thus, let fall the perpendicular *o x*, which bisects A B, (3 III. Euc.) then say, as sine of the angle *x o* B is to ·5, (= ½ A B=*x* B) so is radius to *o* B. (See Trigonometry.) In the trigon, the angle *o* B *x* is 30°, and *x* o B 60°; then



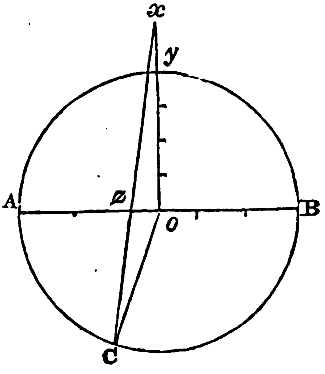
As sine $x\,o$ B 60°	9·937531
Is to x B ·5	1·698970
So is radius 90°	10·000000
	9·698970
	9·937531
To o B ·5773503, ...	—1·761439

Now to find the radius of the circumscribing circle, that is o B, when the side of the triangle is of any given length, as, 10 yards, 10 miles, &c. From the property of similar triangles, 1 : ·5773503 :: 10 (= A B) : 10 × ·5773503 (= o B.) The rest of the tabular numbers may be found in a similar manner. The reason of multiplying the side of any polygon by the number corresponding to it in the third column of the table, may be seen from the last analogy.

PROBLEM XXIII.

In a given circle to inscribe any regular polygon; or, to divide the circumference of a given circle into any number of equal parts.

Divide the diameter A B, into as many equal parts as the figure has sides; erect the perpendicular $o\,x$, from the centre o; divide the radius $o\,y$ into four equal parts, and set off three of these parts from y to x; draw a line from x to the second division z, of the diameter A B, and produce it to cut the circumference at C; join A C, and it will be the side of the required polygon.*

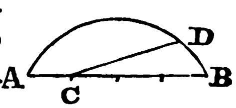


* Let the circumference be denoted by C, and let n denote the number of sides in the polygon; also, let A B be divided into n parts; join C o,

PROBLEM XXIX.

To draw a straight line equal to any given arc, of a circle, A B.

Divide the chord A B into four equal parts; and set off one of these parts from B to D; then join D C, and it will be equal to the length of half the given arc nearly.*

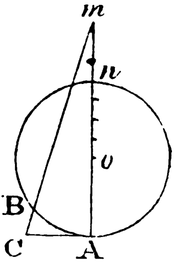


The angle A o C $\left(=\frac{C}{n}\right)$ is given, as also the angle o A C, or o C A $\left(=90-\frac{C}{2n}\right)$ is given; A o $\left(=\frac{n}{2}\right)$ being given, $o\,z$ $\left(=\frac{n}{2}-2\right)$ is given: therefore C z can be found, as also the angle A z C, or its equal $o\,z\,x$; hence the compliment of the angle $o\,z\,x$, viz. $o\,x\,z$ can be found; and $o\,z$ being given, $o\,x$ can be found, and hence $x\,y$ can be found, which will be found to be $\frac{1}{4}$ of $o\,y$ nearly, hence the construction is obvious. Every regular polygon may be either inscribed in a circle, or described about it. But not so of the irregular ones, except the triangle, and another particular case. An equilateral figure inscribed in a circle, is always equiangular; but an equiangular figure inscribed in a circle is not always equilateral, except when the number of sides is odd. For we know from Geometry that if the sides be an even number, then they may either be all equal, or else half of them may be equal, and the other half equal to each other, but different from the former half, the equals being placed alternately. It may further be observed, that as the regular trigon, square, and pentagon can be inscribed geometrically in a circle, and as an arc can be bisected geometrically, it follows that any polygon whose number of sides is expressed by 2^n, $3{\cdot}2^n$, $5{\cdot}2^n$, may be inscribed in a given circle by the scale and compasses only. And were not the method too complicated to be inserted here, we might shew that a polygon, the number of whose sides is a prime number of the form 2^n+1, may also be inscribed geometrically in a circle; a problem which, for many ages, had not been considered possible, till an elegant solution of it was published by M. Gauss, in a work entitled "*Disquisitiones Arithmeticæ.*" For more on this subject, see the analysis of angular Sections, Trigonometry.

* Draw E G at right angles to A B, E being the centre, join E B

Or thus:

From the extremity of the arc A B, whose length is required to be found, draw A o m, passing through the centre; divide o n into four equal parts, and set off three of these parts from n to m; draw m B, and produce it to meet A C drawn at right angles to A m; then will A C be nearly equal in length to the arc A B.*



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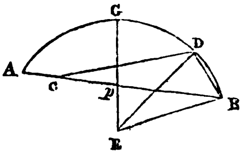
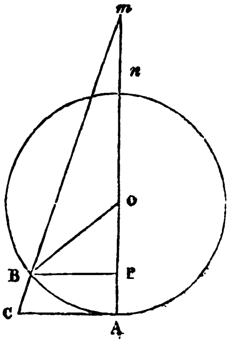
and E D. Let the radius E B and versed sine G p be given; then E p is given; and as E B, E D, and D B are given, the angle E B D can be found; and as E B, B p, and E p, are given, the E B p can be found, and hence the angle C B D can be found. Now as C B, B D, and the angle C B D are given, the side C D can be found, which will be found to be equal to half the arc nearly, as discovered by another method which will be given hereafter.



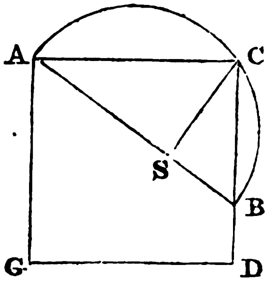
* Let the line A C be assumed equal to the arc A B, join C and the extremity B, of the given arc, and produce C B to meet the production of A o at m.

Now, because the arc is given, its sine B P is given, and also A C: then by similar triangles A C : B P :: A m : P m, and by division A C − P B : B P :: A m − P m (= A P) : P m; and A C − B P, P B, and A P, (this being the versed sine of double the arc) are given; hence P n, and therefore m n can be found, which, when compared with the radius o n, will be $\frac{3}{4}$ of it nearly. Hence the reason of the construction.



PROBLEM XXX.

To make a square equal in area to a given circle.



First divide the diameter A B into fourteen equal parts, and set off eleven of them from A to S; from S erect the perpendicular S C and join A C, the square of which will be very nearly to the area of the given circle.*

PROBLEM XXXII.

To construct a Diagonal Scale.

Draw an indefinite straight line; set off any distance A B according to the intended length of the scale; repeat A E any number of times, E G, G B, &c.; draw C D parallel to A B at any convenient distance; then

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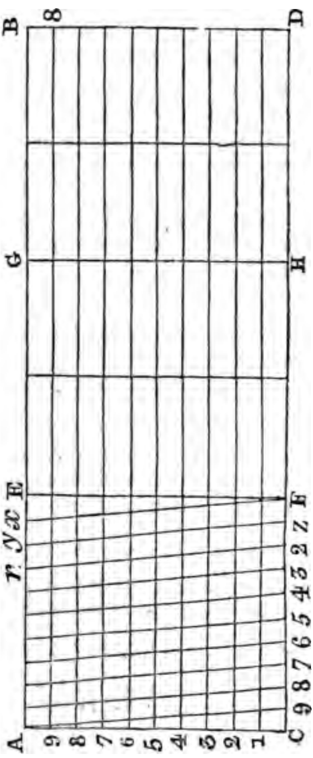
* *Demonstration.* $AB \times AS = AC^2$ (8. VI.); and the diameter of a circle being to its circumference as 7 to 22, nearly, as will be shown hereafter; then $7 : 22 \therefore AB : \frac{AB}{7} \times 22 =$ circumference of the circle whose diameter is A B. It will be shown hereafter, that half the diameter multiplied by half the circumference will give the area of the circle: therefore, $\frac{AB}{7} \times 11 \times \frac{AB}{2} = \frac{14}{7} \times 11 \times \frac{14}{2} = 2 \times 11 \times \frac{14}{2} = 14 \times 11 = AB \times AS = AC^2$. Hence it appears that the square of A C, viz. A C D G, is nearly equal to the area of a circle whose diameter A B contains 14 parts, and the distance A S, from the perpendicular, 11 of such parts. And if 7 and 22 were strictly to each other as the diameter of a circle to its circumference, the square of A C would be the true area of the circle whose diameter is A B; but the ratio of 7 to 22 expresses only the approximate ratio of the diameter to the circumference, these being incommensurable; therefore the square of A C only approaches the area of the circle, differing, however, from the truth only by a very small quantity.

Fig. 1.



draw the perpendiculars A C, EF, G H, B D, &c. Divide A E and A C each into ten equal parts; through 1, 2, 3, &c. draw lines parallel to A B, and through *x*, *y*, &c. draw *x* F, *y* Z, &c. as in the annexed figure.

The principal use of this scale is, to lay down any line from a given measure; or to measure any line and compare it with others.—Whatever number C F represents, F Z will be the tenth of it, and the subdivisions in the vertical direction will be the one-hundredth part. Thus, if CF be units, the small divisions in C F, viz. F Z will be 10ths, and the divisions in the altitude will be the 100th parts of an unit. If C F be tens, the small divisions F Z will be units, and those in the vertical line, tenths; if C F be hundreds, the others will be tens and units. The reason of this is self-evident. Because each division in C F is the tenth of itself, C F being divided into ten equal parts; it follows that if CF be units, F Z, Z 2, &c. will be each 1-10th of an unit; if C F be 10, F Z, Z 2, &c. will be 1. Again, let the triangle E F *x* (Fig. 2,) represent the triangle *E F x (Fig. 1,)* and let E *x* be 1, then by similar triangles

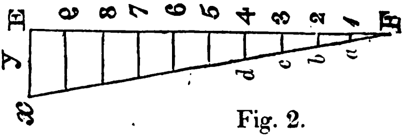


Fig. 2.

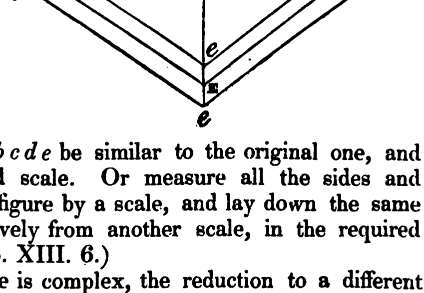
F E (10) : E x (1) :: F 1 (1) : 1 a; that is, 10 : 1 :: 1 : $\frac{1}{10}$ = 1 a. Also, F E (10) : E x (1) :: F 2 (2) : 2 b; that is, 10 : 1 :: 2 : $\frac{2}{10}$ = 2 b. Hence it appears that the three divisions form a continued proportion, the ratio being ten.

To take any number off the scale, as suppose 2 $\frac{38}{100}$, that is, 2·38 : place one foot of the compasses at D, and extend the other to the division marked 3; then move the compasses upward, keeping one foot on the line D B, and the other on the line 3 r, till you arrive at the eighth interval, marked 88, and the extent on the compasses will be that required. This however may express indifferently 2·38, 23·8, or 238, according to the magnitude of the assumed unit.

Note. If C F were divided into 12 equal parts, each division would be 1 inch, and each vertical division 1-10th of an inch, by making C F one foot.

PROBLEM XXXIII.

To reduce a rectilinear figure to a similar one upon either a smaller or larger scale.

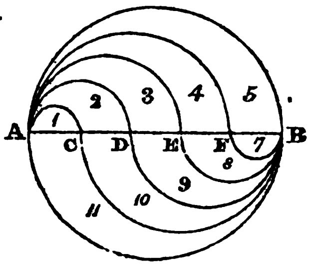


Take any point P in the figure A B C D E, and from this assumed point draw lines to all the angles of the figure; upon one of which take P a to P A in the proposed scales; then draw $a\,b$ parallel to A B, $b\,c$, to B C, &c., then shall the figure $a\,b\,c\,d\,e$ be similar to the original one, and upon the required scale. Or measure all the sides and diagonals of the figure by a scale, and lay down the same measures respectively from another scale, in the required proportion. (Prop. XIII. 6.)

When the figure is complex, the reduction to a different scale is best accomplished by means of the Eidgraph, an instrument invented by Professor Wallace, or by means of the improved Pantograph.

PROBLEM XXXIV.

To divide a circle into any number of equal parts, having their perimeters equal also.



Divide the diameter A B into the required number of equal parts, at the points C, D, E, &c.; then on one side describe the semi-circles 1, 2, 3, 4, &c., and on the other side of the diameter describe the semi-circles 7, 8, 9, 10, &c. on the diameters B F, B E, B D, B C, &c.; so shall the parts 1 11, 2 10, 3 9, 4 8, &c. be equal both in area and perimeter.—Lesley's *Geometry*.

MENSURATION OF SUPERFICIES.

SECTION II.

THE area of any plain figure is the space contained within its boundaries, and is estimated by the number of square miles, square yards, square feet, &c. which it contains.

I. *Long Measure.*				II. *Square Measure.*			
12	Inches	1	Foot.	144	Inches	1	Foot.
3	Feet	1	Yard.	9	Feet	1	Yard.
6	Feet	1	Fathom.	36	Feet	1	Fathom.
$16\frac{1}{2}$	Feet Eng.	1	Pole or	$272\frac{1}{4}$	Feet Eng.	1	Pole or
$5\frac{1}{2}$	Yards		Perch.	$30\frac{1}{4}$	Yards		Perch.
40	Perches	1	Furlong.	1600	Perches	1	Furlong.
8	Furlongs	1	Mile.	64	Furlongs	1	Mile.

In Ireland 21 feet make 1 pole or perch, and 7 yards therefore will make a pole or perch. There are other measures used, for which see *Arithmetical Tables*.

Land is generally measured by a *Chain*, of 4 poles, or 22 yards; it consists of 100 links, each link being ·22 of a yard. —*For further particulars, see Treatise on Surveying.*

Duodecimals are calculations by feet, inches, and parts, which decrease by twelves: hence they take their name.

Multiplication of feet, inches, and parts, is sometimes called cross Multiplication, from the factors being multiplied cross ways. It is used in finding the contents of work done by artificers, where the dimensions are taken in feet, inches, and parts.

RULE I. Under the multiplicand, write the multiplier, taking care to write like denominations under one another; namely, feet under feet, inches under inches, &c.

II. Multiply each term of the Mnltiplicand by the feet in the multiplier, beginning at the lowest denomination in the multiplicand, and set each result under its respective term, observing to carry one for every twelve, from each lower denomination to the next higher.

III. Also multiply every term in the multiplicand by the inches in the multiplier, and set the result of each term one place farther to the right of those in the multiplicand.

IV. Proceed with the rest of the denominations, and set the result of each product one place more towards the right hand than the next preceding, and the sum of all the partial products, will be the whole product required.

IN CROSS MULTIPLICATION.

Feet multiplied by feet, give feet.
Feet by inches, give inches.
Feet by parts, give parts, &c.

Inches by inches, gives parts.
Inches by parts, give thirds.
Inches by thirds, give fourths, &c.

Parts by parts, give fonrths.
Parts by thirds, give fifths.
Parts by fourths, give sixths, &c.

Thirds by thirds, give sixths.
Thirds by fourths, give sevenths.
Thirds by fifths, give eighths.*

* The reason of this rule is almost self-evident. When feet are multiplied by feet, the product will evidently be square feet, when feet are multiplied by inches, the product will give rectangles of a foot long and an inch broad, which must be divided by 12, to give square feet, when feet are multiplied by parts, the product is square inches, or rectangles of a foot long, and one part broad; that is, 12 inches long, and $\frac{1}{12}$ of an inch broad, which are equal to rectangles of a square inch each. By a similar mode of reasoning the other denominations are known. When feet are concerned, the product is of the same name with the terms multiplied by the feet, but *when feet are not* concerned, the product is expressed by the sum of *the indices of the factors.*

1. Multiply 7 feet 9 inches by 3 feet 6 inches.

```
   F.  I.
   7 : 9
   3 : 6
  --------
  23 : 3
   3 : 10 : 6
  -----------
  27 :  1 : 6   Ans.
```

```
                 F.     I.   P.
2. Multiply 240 : 10 : 8 by 9 : 4 : 6
              9 :  4 : 6
          --------------
         2168 :  0 : 0
           80 :  3 : 6 : 8
           10 :  0 : 5 : 4
         ------------------
         2258 :  4 : 0 : 0   Ans.
```

	F. I. P.		F. I. P.		F. I. P. ‴. ⁗.
3. Multiply	8: 5	by	4: 7.	*Ans.*	38: 6:11.
4. Multiply	9: 8	by	7: 6.	—	72: 6.
5. Multiply	7: 6	by	5: 9.	—	43: 1: 6.
6. Multiply	4: 7	by	3:10.	—	17: 6:10.
7. Multiply	7: 5:9	by	3: 5:3.	—	25: 8: 6:2:3.
8. Multiply	10: 4:5	by	7: 8:6.	—	79:11: 0:6:6.
9. Multiply	75: 7:0	by	9: 8:0.	—	730: 7: 8.
10 Multiply	57: 9:0	by	9: 5:0.	—	543: 9: 9.
11. Multiply	75: 9:0	by	17: 7:0.	—	1331:11: 3.
12. Multiply	321: 7:3	by	9: 3:6.	—	2988: 2:10:4:6.
13. Multiply	4: 7:8	by	9: 6.	—	44: 0:10.
14. Multiply	39:10:7	by	18: 8:4.	—	745: 6:10:2:4.

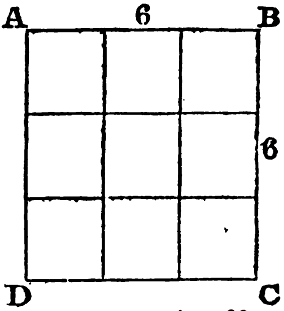
Note. All these can be solved by the method of aliquot parts, thus:—

15. Multiply 368 F. : 7′ : 5″ by 137 F. : 8′ : 4″

		137 : 8 : 4
		2576
		1104
		368
$6' = \frac{1}{2}$	..	184 : 3 : 8 : 6
$2' = \frac{1}{3}$	..	61 : 5 : 2 : 10
$4'' = \frac{1}{6}$	..	10 : 2 : 10 : 5 : 8
$6' = \frac{1}{2}$	..	68 : 6
$1' = \frac{1}{6}$	..	11 : 5
$4'' = \frac{1}{3}$	..	3 : 9 : 8
$1'' = \frac{1}{4}$	..	0 : 11 : 5
Ans.		50756 : 7 : 10 : 9 : 8

PROBLEM I.

To find the area of a Square.



RULE. Multiply the side by itself, and the product will be the area.

1. Let the side of the square ABCD be 6; what is its area?
Ans. $6 \times 6 = 36$, the area.

The reason of this is too evident to require a demonstration.

2. What is the area of a square whose side is 15 chains?
Ans. 225.

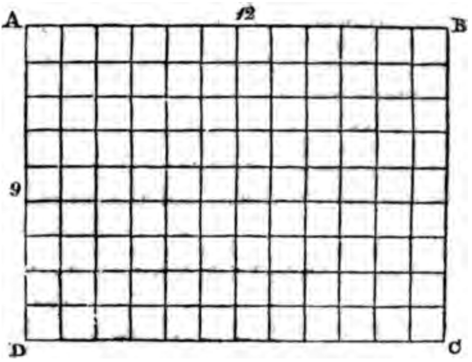
3. What is the area of a square whose side is 7 feet 9 inches? *Ans.* $60\frac{1}{16}$.

4. What is the area of a square whose side is 4769 links?
Ans. 25643361.

PROBLEM II.

To find the area of a rectangle.

RULE. Multiply the length of the rectangle by its breadth, and the product will be the area.



1. Let the sides of the rectangle A B C D be 12 and 9, what is its area? *Ans.* $12 \times 9 = 108$, the area. The reason of this is evident.

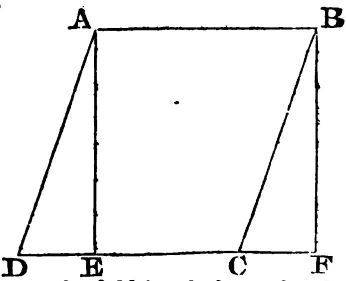
2. What is the superficial contents of a plank, whose length is 5 feet 6 inches, and breadth 7 feet 8 inches?
Ans. 42 feet 2 inches.

3. What is the area of a field whose boundaries form a rectangle, its length being 176 links and breadth 154 links?
Ans. ·27104 of an acre.

4. What is the superficial content of a floor, whose length is 40 feet 6 inches, and breadth 28 feet 9 inches?
Ans. 1164 feet 4 inches 6 parts

PROBLEM III.

To find the area of a Rhombus.



RULE. Multiply the length by the perpendicular breadth, and the product will be the area.*

1. What is the area of a rhombus, whose side is 16 feet, and perpendicular breadth 10 feet? *Ans.* 16 × 10 = 160 feet, the area.

2. What is the contents of a field in theform of a rhombus, whose length is 7·6 chains, and perpendicular height 5·7 chains? *Ans.* 43·32 chains.

3. What is the area of a rhombus, whose side is 7 feet 6 inches, and perpendicular height 3 feet 4 inches? *Ans.* 25 feet.

4. What is the area of a rhombus, in square yards, whose length is 3 yards, and perpendicular height 2 feet 3 inches? *Ans.* 20 feet 3 inches.

PROBLEM IV.

To find the area of a Triangle.

RULE. Multiply the base by the perpendicular height, and divide the product by two for the area.†

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* *Demonstration.* Because the parallelogarms A B C D, and A B F E, are equal (35 : 1); but the area of A B F E, is found (Problem II.) by multiplying its length and breadth; that is, the area of A B F E, is equal to A B×B F=D C × A E, which is the rule.

The continual product of any two sides of a parallelogram, and the natural sign of their contained angle will give the area; that is A D × D C × by the natural sign of the angle D, will give the area. For the demonstration, see *Trigonometry.*

† *Demonstration.* The product of the base and perpendicular

1 The base of a triangle is 76·5 feet, and perpendicular 92·2 feet; what is its area?

Ans. $76{\cdot}5 \times 92{\cdot}2 \div 2 = 3526{\cdot}65$ square feet, the area.

2. The base of a triangle is 72·7 yards, and the perpendicular height 36·5 yards; what is its area?

Ans. 1326·775 yards.

3. The base of a triangular field is 1276 links, and perpendicular 976 links; how many acres in it?

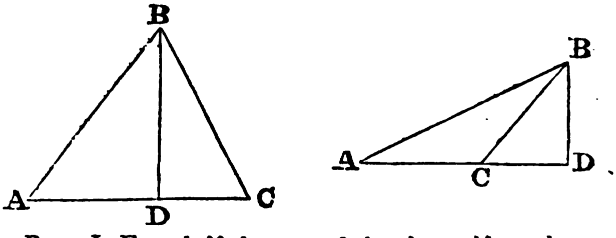
Ans. 6 acres 1 rood 16·3008 perches.

4. The base of a triangle measures 15 feet 6 inches, and the perpendicular 12 feet 7 inches; what is its area?

Ans. 97 feet 6¼ inches.

PROBLEM V.

Having the three sides of any Triangle given, to find its area.



Rule I. From half the sum of the three sides subtract each side separately, then multiply the half sum and the three remainders together, and the square root of the last product will be the area of the triangle.*

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height gives the area of a rectangular parallelogram, whose sides are equal to the base and perpendicular of the triangle, (Prob. II.); but the triangle being half of the parallelogram, (41. I.) it follows that half the product of the base and perpendicular will give its area.

* *Demonstration.* Put $AC = a$, $AB = b$, $BC = c$, and let half the sum of the sides be denoted by S, that is, $S = \frac{AB + BC + AC}{2}$, and T equal the area of the triangle; then, $2\,AC \times CD = a^2 + c^2 - b^2$ (26. II.); therefore $CD = \frac{a^2 + c^2 - b^2}{2a}$. Again, $BD^2 =$

RULE II. Divide the difference between the squares of two sides of the triangle by the third side; to half this third side add half the quotient, and deduct the square of this sum from the square of the greater side, the remainder will be the square of the perpendicular, the square root of which, multiplied by half the base, will give the area of the triangle*.

1. Given the side $AB = 9{\cdot}2$, $BC = 7{\cdot}5$, and $AC = 5{\cdot}5$. Required the area of the triangle?

$$\begin{array}{r} 9{\cdot}2 \\ 7{\cdot}5 \\ 5{\cdot}5 \\ \hline \text{Sum } 22{\cdot}2 \\ \hline \end{array}$$

$$\left.\begin{array}{r} \tfrac{1}{2}\text{ Sum } 11{\cdot}1 - 9{\cdot}2 = 1{\cdot}9 \\ 11{\cdot}1 - 7{\cdot}5 = 3{\cdot}6 \\ 11{\cdot}1 - 5{\cdot}5 = 5{\cdot}6 \end{array}\right\} \text{; then } \sqrt{(11{\cdot}1 \times 1{\cdot}9 \times 3{\cdot}6 \times 5{\cdot}6)} = \sqrt{425{\cdot}1744} = 20{\cdot}619 \text{ the area by RULE I.}$$

$BC^2 - CD^2$ (47·1) $= c^2 - \left(\dfrac{a^2 + c^2 - b^2}{2a}\right)^2$; but (Prob. IV.) $T = \dfrac{AC \times BD}{2}$; therefore $T^2 = \dfrac{AC^2 \times BD^2}{4} = \dfrac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{16}$; but this expression, consisting of the difference of two squares, $\left(\dfrac{4a^2c^2}{4} \text{ and } \dfrac{(a^2 + c^2 - b^2)^2}{4}\right)$ which is equal to the rectangle under their sum and difference, (Cor. 5. II.) may be transformed to the following expression, viz. $T^2 = \dfrac{2a + a^2 + c^2 - b^2}{4} \times \dfrac{2ac - a^2 - c^2}{4}$ $= \dfrac{(a + c)^2 - b^2}{4} \times \dfrac{b^2 - (a - c)^2}{4}$; and by decomposing these factors again, we get $T^2 = \dfrac{a + b + c}{2} \times \dfrac{a - b + c}{2} \times \dfrac{-a + b + c}{2} \times \dfrac{-a + b + c}{2}$. From the assumption, $S = \dfrac{a + b + c}{2}$, $S - b = \dfrac{a - b + c}{2}$, $S - c = \dfrac{a + b - c}{2}$, and $S - a = \dfrac{-a + b + c}{2}$. Hence, by substitution $T^2 = S \times (S - a) \times (S - b) \times (S - c$; and $T = \sqrt{(S \times (S - a) \times (S - b) \times (S - c),)}$ which is the rule. For a Geometrical demonstration of this, see *Trigonometry*.

If two sides of any triangle, and the contained angle be given, the area may be found by multiplying the sides together, and the product *by the natural* sine of the contained angle, and dividing the result *by 2. For the reason, see Trigonometry.*

* *This is a corollary to the* 6. II.

$9{\cdot}2^2 - 7{\cdot}5^2 = 84{\cdot}64 - 56{\cdot}25 = 28{\cdot}39$; then $28{\cdot}39 \div 5{\cdot}5 = 5{\cdot}161818$ quotient.

Now $5{\cdot}161818 \div 2 + 5{\cdot}5 \div 2 = 2{\cdot}580909 + 2{\cdot}75 = 5{\cdot}3309$ = half quot. plus half third side: then $84{\cdot}64 - 28{\cdot}41849481 = 56{\cdot}22150519$, and $\sqrt{56{\cdot}22150519} \doteq 7{\cdot}498$ = perpendicular; hence $7{\cdot}498 \times 2{\cdot}75 = 20{\cdot}619$ the area as before.

2. What is the area of a triangle whose sides are 50, 40, and 30? *Ans.* 600.

3. The sides of a triangular field are 4900, 5025, and 2569 links; how many acres does it contain?
Ans. 61 acres 1 rood 39·68 perches.

4. What is the area of an isosceles triangle, whose base is 20, and each of its equal sides 15? *Ans.* 111·803.

5. How many acres are there in a triangle, whose three sides are 380, 420, and 765 yards?
Ans. 9 acres 0 rood 38 poles.

6. How many square yards in a triangle, whose three sides are 13, 14, and 15 feet? *Ans.* $9\frac{1}{3}$ square yards.

7. How many acres, &c. in a triangle, whose three sides are 49, 50·25, and 25·69 chains?
Ans. 61 acres 1 rood 39·68 perch.

PROBLEM VI.

To find the area of an equilateral triangle.

RULE. Square the side, and from this square deduct its fourth part; then multiply the remainder by the fourth part of the square of the side, and the square root of the product will give the area.* Or mnltiply $\frac{AB^2}{4}$ by $\sqrt{3}$ for the area.†

* *Demonstration.* $CB^2 - BD^2 = CD^2$ (47. I.) but $BD^2 = \frac{CB^2}{4}$, (Cors. 26. I. and 4. II.) therefore $\left(CB^2 - \frac{CB^2}{4}\right) \times \frac{CB^2}{4} =$ the square of the area as may be derived from Prob. III. Hence $\sqrt{\left((CB^2 - \frac{CB^2}{4}) \times \frac{CB^2}{4}\right)}$ = area of the triangle which is the rule.

† By *the first* part of the rule, the area of an equilateral triangle,

1. Each side of a triangular field, A B C, measures 4 perches, what is its area?

Ans. $4^2 = 16$; then $16 \div 4 = 4\cdot$ and $16 - 4 = 12$: then $12 \times \frac{16}{4} = 12 \times 4 = 48$, and $\sqrt{48} = 6\cdot928$ the area.

2. How many acres in a field of a triangular form, each of whose sides measures 70 perches?

Ans. 13 acres 0 rood 39·25 perches.

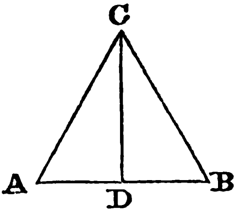
3. The perimeter of an equilateral triangle is 27 yards, what is its area? *Ans.* 140·13.

Note. When the triangle is isosceles, the perpendicular is equal to the square root of the difference between the squares of either of the equal sides and half the base.

PROBLEM VII.

Given the area and altitude of a triangle, to find the base.

RULE. Divide the area by the altitude or perpendicular, and double the quotient will give the base*.



1. Given the area of a triangle $= 12$ yards, and altitude $= 4$; what is its base?

Ans. $12 \div 4 = 3$; then $3 \times 2 = 6$ yards, the base A B.

2. A Surveyor having lost his field-book, and requiring the base of a triangular field, whose contents he knew from recollection was 14 acres and altitude 7 yards, how much is the base? *Ans.* 19360 yards.

---

each of whose sides measures 1, is $\frac{\sqrt{3}}{4}$; and similar triangles being to each other as the squares of their homologous sides, (19. VI.) we have $1^2 : AB^2 :: \frac{\sqrt{3}}{4} : \frac{AB^2}{4} \sqrt{3}$.

Cor. As similar polygons are to each other in the duplicate ratio of their homologous sides, (20. VI.) it follows that the square of the side of any polygon, multiplied by the area of a similar polygon whose homologous side is 1, will give the area of the given polygon.

* *Demonstration.* $CD \times \frac{AB}{2} =$ area (Prob. VI.) $\because \frac{AB}{2} = \frac{\text{area}}{CD}$, and $AB = \frac{\text{area}}{CD} \times 2$.

PROBLEM VIII.

Given the area of a triangle, and its base, to find its altitude.

RULE. Divide the area by the given base, and double the quotient will give the perpendicular.

The reason of this rule is manifest from the last.

1. Given the area of a triangle $= 12$, and its base $= 6$, what is its perpendicular height?

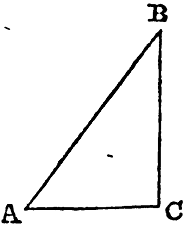
Ans. $12 \div 6 = 2$; then $2 \times 2 = 4$ the altitude.

PROBLEM IX.

Given any two sides of a right angled triangle, to find the third side, and thence its area.

RULE I. To the square of the perpendicular add the square of the base, and the square root of the sum will give the hypothenuse.

II. The square root of the difference of the squares of the hypothenuse and either side, will give the other. Or multiply the sum of the hypothenuse and either side, by their difference; and the square root of the product will give the other.*



1. Given the base A C 3, the perpendicular C B 4; required the hypothenuse A B?

Ans. $3^2 + 4^2 = 25$; then $\sqrt{25} = 5$, the hypothenuse A B.

2. Given A B 5, A C 3; required C B?

Ans. $5^2 - 3^3 = 16$; then $\sqrt{16} = 4$ the side B C; or, $(5+3) \times (5-3) = 8 \times 2 = 16$; then $\sqrt{16} = 4$, as before.

* The truth of this rule is evident from 47. I. and from Cor. 5. II., which says, that the rectangle under the sum and difference of two quantities *is equal to* the difference of their squares.

3. Given A B : 5, B C 4; required A C : $5^2 - 4^2 = 9$; then $\sqrt{9} = 3$ the side A C; or $(5 + 4) \times (5 - 4) = 9 \times 1 = 9$; then $\sqrt{9} = 3$, as before. And $3 \times 4 \div 2 = 6$ the area of the triangle.

4. The wall of a building on the brink of a river is 120 feet, and the breadth of the river is 70 yards, what is the length of the chord in feet that will reach from the top of the building across the river? *Ans.* 243.14 feet.

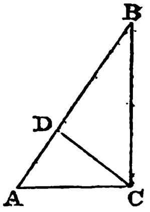
5. A ladder 60 feet long, will reach to a window 40 feet from the flags on one side of a street, and by turning the ladder over to the other side of the street, it will reach a window 50 feet from the flags; required the breadth of the street? *Ans.* 77·8875 feet.

6. The roof of a house forms a right angle at the top, the length of one rafter being 10 feet and its opposite one 14 feet, what is the breadth of the house? *Ans.* 17·204.

PROBLEM X.

Given the base and perpendicular, to find the perpendicular let fall on the hypothenuse from the right angle; and also the segments into which the hypothenuse is divided by this perpendicular.

Rule. Find the hypothenuse by Prob. IX. Then divide the square of the greater side by the hypothenuse, and the quotient will give the greater segment, which deducted from the entire will give the less. Having found the segments, multiply them together, and the square root of the product will give the perpendicular.*



* *Demonstration.* By the 8. VI. $AB \times BD = BC^2$. Hence, $BD = \frac{BC^2}{AB}$, which is one part of the rule. Again, $AD \times DB = DC^2$, (8. VI.) hence $DC = \sqrt{AD \times DB}$, which is the second part.

1. Given A C 3 yards, and C B 4 yards; required the segments B D, D A, and the perpendicular D C.

$3^2 + 4^2 = 25$: then $\sqrt{25} = 5 =$ A B.

$4^2 \div 5 = 16 \div 5 = 3{\cdot}2 =$ B D; then $5 - 3{\cdot}2 = 1{\cdot}8$ $=$ A D.

Again, $3{\cdot}2 \times 1{\cdot}8 = 5{\cdot}76$; then $\sqrt{5{\cdot}76} = 2{\cdot}4 =$ D C.

2. The roof of a house whose walls are 30 feet high, forms a right angle at the top, now if one of the rafters be 10 feet long, and its opposite yoke-fellow 12; required the breadth of the building, the length of the prop set upright to support the ridge of the roof, and the part of the floor at which it must be placed?

Ans. Breadth of the building 15·6204 feet, greater segment 9·2123 feet, less segment 6·4081 feet, and length of the prop 37·68 feet.

3. The side wall of a house is 30 feet high, and its opposite one 40, the roof forms a right angle, as in the last Problem; the length of the rafters is the same, viz. 10 feet and 12; the end of the shorter is placed on the higher wall, and *vice versa*; required the length of the upright which supports the ridge of the roof, and the breadth of the house?

Ans. 41·8048, length of upright, and 12 feet the breadth of the house.

4. One of the side walls of a building is 30 feet high; its roof is to be the same, in every respect, as in the last Problem, and its top is to be over the middle of the building; what must the height of the other wall be, and the breadth of the building to answer the conditions; the end of the 12 foot rafter resting on the 30 foot wall?

Ans. 31·9956 height of wall, and 15·484 the breadth of the house.

PROBLEM XI.

To find the area of a Trapezium.

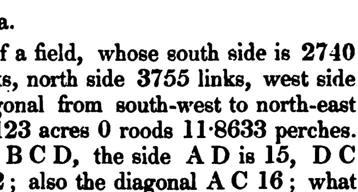
RULE. Divide the trapezium into two triangles, by joining two of its opposite angles; find the area of each triangle; and the sum of both areas will give the area of the trapezium.

Or:

Draw two perpendiculars from the opposite angles to the diagonal; then multiply the sum of these perpendiculars by the diagonal, and half the product will give the area.*

1. In the trapezium A B C D, the diagonal A C is 100 yards, the perpendicular D E 35, and E F 30; what is its area?

$$\begin{array}{r} DE = 35 \\ EF = 30 \\ \hline 65 \\ 100 \\ \hline 2)6500 \\ \hline 3250 \end{array}$$ the area.



2. What is the area of a field, whose south side is 2740 links, east side 3575 links, north side 3755 links, west side 4105 links, and the diagonal from south-west to north-east 4835 links? *Ans.* 123 acres 0 roods 11·8633 perches.

3. In the trapezium A B C D, the side A D is 15, D C 13, C B 14, and A B 12; also the diagonal A C 16; what is its area? *Ans.* 172·5247.

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* *Demonstration.* By Prob. IV. the area of the triangle A B C $= \frac{AC \times BF}{2}$, and that of the triangle A D C $= \frac{AC \times DE}{2}$. Then the sum of these two areas will be the area of the trapezium, viz.— $\frac{AC \times BF}{2} + \frac{AC \times DE}{2} = \frac{BF + DE}{2} \times AC$, which is the rule.

4. In the trapezium A B C D, there are given A B 220 yards, D C 265 yards, and A C 378 yards; also A F 100 yards, and E C 70 yards; what is its area?

Ans. 85342·2885 yards = 17 acres 2 roods 21 perches.

5. In the trapezium A B C D, there are given A B 220 yards, DC 265 yards, B F 195·959 yards, D E 255·5875 yards; also F E 208 yards; required the area of the trapezium? *Ans.* 85342·2885 yards.

6. Suppose in the trapezium A B C D, on account of obstacles, I can only measure A B, D C, B F, D E, and F D, which are respectively 22 yards, 26 yards, 19 yards, 25 yards, and 32 yards, required the area?

Ans. 840·62 square yards.

PROBLEM XII.

To find the area of a trapezium inscribed in a circle, or of any one whose opposite angles are together equal to two right angles.

RULE. Add the four sides together, and take half the sum, from this half sum deduct each side separately; and the square root of the product of the four remainders will give the area of the trapezium.*

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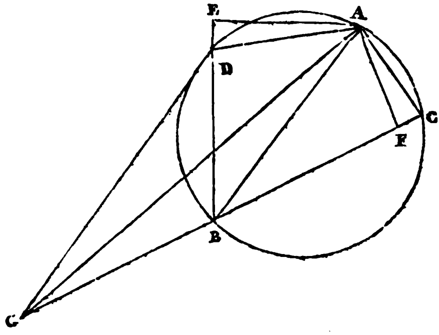
* *Demonstration.* Let A C $= a$, C B $= b$, B D $= c$, and A D $= d$; also, let the diagonal A B $= x$. Draw D G parallel to A B, meet-



1. What is the area of a four-sided field, whose opposite angles are together equal two right angles, the length of the four sides being as follows, viz. A B 12·5, A C 17, D C 47·5, and B C 8 yards?

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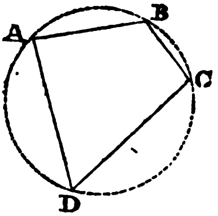
ing C B produced in G; join G A; and from A draw the perpendiculars A F (p) A E (p'). Because D G is parallel to A B, the triangles A B D and A B G are equal, (37. I.) to each add the triangle A B C, then the triangle A G C is equal to the quadralateral A D B C inscribed in the circle. Now as the triangles A G B, A D B are equal, their bases G B, B D are reciprocally proportional to their perpendiculars p and p'; that is, D B : B G :: $p : p'$. But the triangles A D E, A F C are similar, being right-angled at E and F, and the angles A D E and A C F supplements of the angle A D B; therefore (4. VI.) $p : p' :: a : d \therefore c : \mathrm{BG} :: a : d$: hence $\mathrm{GB} = \frac{cd}{a}$, and $x^2 = a^2 + b^2 - 2b \times \mathrm{CF}$ (13. II.); also, $x^2 = c^2 + d^2 + 2c \times \mathrm{DE}$ (12. II.); therefore, $\mathrm{CF} = a^2 + b^2 - (c^2 + d^2) - 2c \times \mathrm{DE}$; but $a : d :: \mathrm{CF} : \mathrm{DE} \because \mathrm{DE} = \frac{d \times \mathrm{CF}}{a}$; then by substitution and transposition, we get $\mathrm{CF} = \frac{a}{2(ab+cd)} \times (a^2 + b^2 - c^2 - d^2)$; but $p = \sqrt{(a^2 - \mathrm{CF}^2)} = \left\{a^2 - a^2\left(\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right)\right\}^{\frac{1}{2}}$; $p = \frac{a}{2(ab+cd)}\left\{(2ab+2cd)^2 - (a^2+b^2-c^2-d^2)^2\right\}^{\frac{1}{2}}$; but $\frac{p \times \mathrm{CG}}{2}$, or $\frac{p}{2}\left(b + \frac{cd}{a}\right)$ or $\left(\frac{ab+cd}{2a}\right)p$ = the area = (by substitution) $\frac{1}{4}\left\{(2ab+2cd)^2 - (a^2+b^2-c^2-d^2)^2\right\}^{\frac{1}{2}}$

$$= \frac{1}{4}\left\{(2ab+2cd+a^2+b^2-c^2-d^2) \times (2ab+2cd+a^2 - b^2+c^2+d^2)\right\}^{\frac{1}{2}} = \frac{1}{4}\left\{(a^2+2ab+b^2-c^2+2cd-d^2) \times (-a^2+2ab-b^2+c^2+2cd+d^2)\right\}^{\frac{1}{2}} = \frac{1}{4}\left\{[(a+b)^2-(c-d)^2] \times [-(a-b)^2+(c+d)^2]\right\}^{\frac{1}{2}} = \frac{1}{2}\left\{(a+b+c-d) \times (a+b-c+d) \times (a-b+c+d) \times (-a+b+c+d)\right\}^{\frac{1}{2}}$$

$$= \left\{\left(\frac{a+b+c+d}{2} - d\right) \times \left(\frac{a+b+c+d}{2} - c\right) \times\right.$$

12·5
17
17·5
8
——
2)55
——
27·5



$$\left.\frac{a+b+c+d}{2}-b\right)\Big\} \times \left(\frac{a+b+c+d}{2}-a\right)\Big\}^{\frac{1}{2}} = \text{(putting}$$

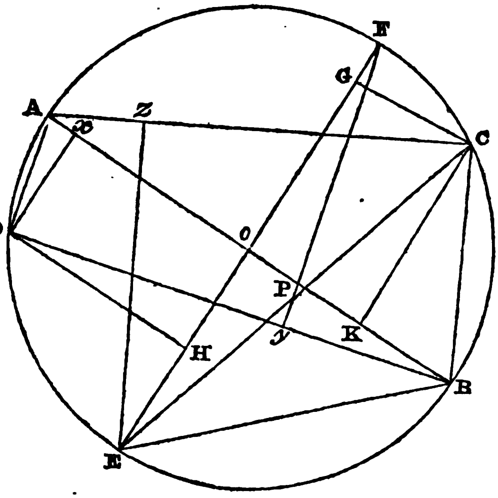
$$S = a+b+c+d)\ \Big\{\left(\frac{S}{2}-d\right)\times\left(\frac{S}{2}-c\right)\times\left(\frac{S}{2}-b\right)\times\left(\frac{S}{2}-a\right)\Big\}^{\frac{1}{2}}$$

which is the rule.

For another Demonstration, see *Trigonometry*.

This may be demonstrated on Geometrical principles, thus:—

Let D A C B be the inscribed trapezium; bisect the angle A C B



by the line C E, and from the point E demit the perpendicular E o F on the line A B, which will evidently bisect it, and also be the dia-

27·5	27·5	27·5	27·5
12·5	17	17·5	8

$15 \times 10·5 \times 10 \times 19·5 = 30612·50$; then $\sqrt{30612·50} = 174·964$, the area in yards.

---

meter of the circle; join E A, E B, and A F; from C let fall the perpendicular C G, and from E, the perpendicular Ez on the line A C; then zC is half the sum of the sides A C, C B, and A z is half their difference. For, the triangles C A P and B E P are similar, having the angles at P equal, and also the angles A C P and P B E equal: then (4. 6.) C A : A P :: B E or A E : E P.

Again, the triangles z A E and o P E are similar; for the angles E A z and E F C are equal, as they stand on the same arc C B E, and the angles A z E and F C E are equal, being right: therefore the triangles A z E and E F C are similar; but the triangles E o P and E F C are evidently similar: therefore the triangles A z E and E o P are similar: hence A z : o P :: A E : E P; but A E : E P :: C A : A P; therefore by similiarity of ratios C A : A P :: A z : o P. Again, because the angle A C B is bisected by the line C E, A C : C B :: A P : P B (3. 6.), and by composition A C + C B : A C :: A P + P B (= A B) : A P; and by alternation, conversion, and division, A C + C B : A P + P B (= A B) :: A C − C B : A P − P B :: ½ (A C − C B) : ½ (A P − P B). But it has been shewn that A C + C B : A C :: A P + P B : A P; then by alternation, A C + C B : A P + P B :: A C : A P : ½ (A C − C B) : ½ (A P − P B) (= o P) :: A C : A P; but it was proved that A C : A P :: A z : o P : therefore ½ (A C − C B) : o P :: A z : o P; hence ½ (A C − C B) = A z. Now as A z is half the difference between the lines A C, and C B, it follows that C z is half the sum of A C and C B. For the same reason D y is half the sum of the lines A D and D B, and B y half their difference: therefore C z + D y = half the sum of the four lines A C, C B, B D, D A; that is, C z + D y = half the perimeter of the inscribed trapezium A C B D. Then if from C z + D y each side of the trapezium be subtracted, respectively, the four remainders will evidently be D y − A z, D y + A z, C z − B y, C z + B y; and which being multiplied together, will give (D y^2 − A z^2) × (C z^2 − B y^2).

But A z^2 (the square of half the difference) = F G × o E. For the triangles A z E and F G C being similar A z^2 : z E^2 :: F G^2 : G C^2 :: F G^2 : F G × G E :: F G : G E. And A z^2 + z E^2 (= A E^2) : A z^2 :: F G + G E (= F E) : F G :: F E × o E : F G × o E; but F E × o E = A E^2, F A E being a right angle; therefore F G × o E = A z^2.

Again, C z^2 (half sum) = F o × G E. For it is obvious that C z^2 = C E^2 − E A^2 + A z^2 = G E × F E − o E × F E + o E × F G = G E × F E − G E × o E = G E × (F E − o E) = G E × F o.

In a similar manner it may be proved that B y^2 (the half differ-

2. There is a trapezium whose opposite angles are together equal to two right angles; the sides are as follows, viz. A B 25, A D 34, D C 35, and B C 16; required its area?

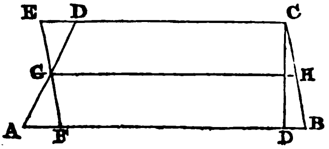
Ans. 700·99.

PROBLEM XIII.

To find the area of a Trapezoid

RULE. Multiply half the sum of the two parallel sides by the perpendicular distance between them, and the product will give the area.*

1. Let A B C D be a trapezoid, the side A B = 40, D C = 25, C D = 18; required the area?



40
25

Ans. $65 \div 2 = 32{\cdot}5 \times 19 = 585$ area.

---

ence) $= F o \times E H$, and $D y^2$ (the half sum) $= F H \times E o$; therefore $(D y^2 - A z^2) \times (C z^2 - B y^2)$, the continual product of the four remainders, $= (F H \times E o - F G \times E o) \times (G E \times F o - E H \times F o) = ((F H - F G) \times E o) \times ((G E - E H) \times F o = G H \times E o \times G H \times F o = G H^2 \times E o \times F o = G H^2 \times o B^2 =$ square of the area of the trapezium; because $C K \times o B =$ area of the triangle A B C, and $D x \times o B =$ area of the triangle A D B; then $(C K + D x) \times o B =$ area of the trapezium, but $C K + D x = H G$; therefore $H G^2 \times o B^2 =$ the square of the trapezium A C B D, which is the rule.

* *Demonstration.* Bisect A D in G; through G draw E F parallel to C B, and G H parallel to A B; produce C D to E. The triangles A G F and E G D are evidently equi-angular, and the sides A G and G D equal; therefore these triangles are equal. (26. I.) Hence the trapezoid A B C D is equal to the parallelogram F E C B, but the area of the parallelogram is equal $F B \times C D$. (Prob. III.) Again, it has been shown that the triangles E D G and A G F are equal in every respect; hence $E D = A F$, and $E C = F B = G H$, (34. I.) therefore G H is half the sum of E C and F B; that is, $G H = \frac{E C + F B}{2} = \frac{D C + A B}{2}$; wherefore $\frac{D C + A B}{2} \times C D$ gives the area of the trepezoid,

2. What is the area of a trapezoid, whose parallel sides are 750 and 1225 links, and the perpendicular height 1540 links? *Ans.* 15 acres 0 roods 33·2 perches.

3. What is the area of a trapezoid whose parallel sides are 4 feet 6 inches, and 8 feet 3 inches; and the perpendicular height 5 feet 8 inches? *Ans.* 36 feet $1\frac{1}{2}$ inches.

4. What is the area of a trapezoid whose parallel sides are 1476, and 2073 yards, and perpendicular height 976 yards? *Ans.* 220 acres 3 roods 25 perches 7 yards.

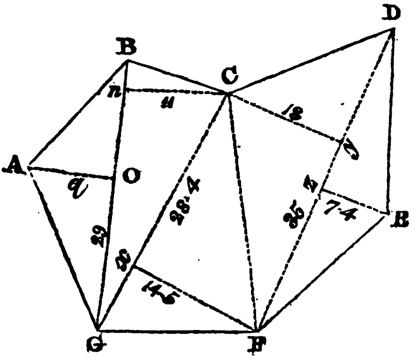
PROBLEM XIV.

To find the area of an irregular Polygon.

RULE. Divide the figure into triangles and trapeziums, and find the area of each separately, by Problem IV. or XI. Add these areas together, and the sum will be the area of the polygon.

1. What is the area of the irregular polygon A B C D E F G A, following lines being given?

A O = 9
G B = 29
C n = 11
G C = 28·4
F x = 14·5
C y = 13
F D = 35
E z = 7·4



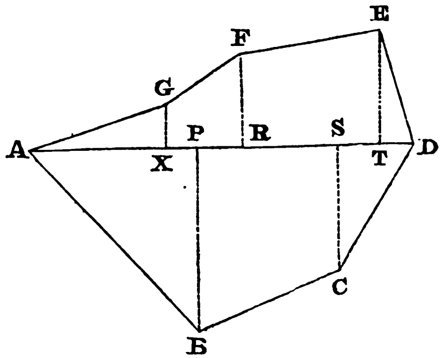
$$
\begin{array}{rl}
AO = 9 & \\
Cn = 11 & \\
\hline
2)20 & \text{sum} \\
\hline
10 & \text{half} \\
29 & \text{diag. } GB \\
\hline
290 & = \text{area of } ABCGA. \\
Cy = 13 & \\
Ez = 7{\cdot}4 & \\
\hline
2)20{\cdot}4 & \text{sum} \\
\hline
10{\cdot}2 & \text{half} \\
35 & \\
\hline
357{\cdot}0 & = \text{area of } FCDEF. \\
Fx = 14{\cdot}5 & \\
14{\cdot}2 & \\
\hline
205{\cdot}9 & = \text{area of } GFC.
\end{array}
$$

$$
\begin{array}{rl}
290 & = \text{area of } ABCGA \\
357 & = \text{area of } FCDEF \\
205{\cdot}9 & = \text{area of } GFC \\
\hline
\textit{Ans. } 852{\cdot}9 & = \text{area of } ABCDEFGA.
\end{array}
$$

2. In a five-sided field G C D E F G there is G C = 28· perches, F x = 14· perches, C y = 13 perches, z E = 7· perches, and F D = 35 perches; required its area?

Ans. 3 acres 1 rood 26 perches.

Note. In finding the area of an irregular figure, draw a line through the extreme angles of the figure, on which let fall perpendiculars from all the other angles of the polygons, which will divide it into triangles and trapezoids; then find the area of these by Problems IV. and XIII.



3. In the annexed figure, there are given in perches,

AX = 15	AP = 17	FR = 10
XR = 8	PS = 14	ET = 12
RT = 14	SD = 95	BP = 20
TD = 6	OX = 5	CS = 14

Required the area? *Ans.* 4 acres 2 roods $17\frac{1}{2}$ perches.

PROBLEM XV.

To find the area of a regular Polygon.

RULE I. Add all the sides together, and multiply half the sum by the perpendicular drawn from the centre of the polygon to the middle of one of the sides, and the product will give the area. This perpendicular is the radius of the inscribed circle.

RULE II. Multiply the square of the side of the polygon by the number standing opposite to its name in the following table, under the word area, and the product will give the area of the polygon.

RULE III. Multiply the side of the polygon, by the number standing opposite to its name in the following table, under *the word perpen*dicular, and the product will be the perpen-

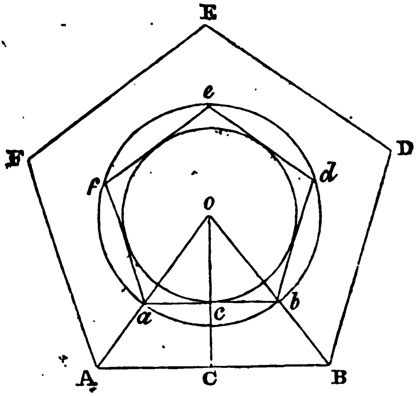
dicular from the centre of the polygon to the middle of one of its sides; then multiply half the sum of the sides by this perpendicular, and the product will give the area.*

---

* *Demonstration.* 1. Every regular polygon is made up of as many triangles as the figure has sides; therefore the area of one of the triangles multiplied by the number of sides, will give the area of the polygon. But the area of one of the triangles is found by multiplying the perpendicular by half the base, (Prob. IV.) therefore the area of the whole polygon is equal to the product of the perpendicular and half the sum of the sides.

2. Regular polygons having the same number of sides, are similar to each other; and similar figures being to each other as the square of their like sides, (20. VI.); therefore, as the areas in the Table are those of polygons whose sides are 1, it follows that 1^2 is to the area in the Table :: the square of the side of the polygon : its area.

3. The perpendiculars from the centres will divide the similar triangles of two similar polygons into similar right angled triangles, one of the homologous or like sides being the perpendicular; therefore 1 : perpendicular in the Table :: the side of the polygon : its perpendicular.



The polygons A B D E F, and *a b d e f* are similar in every respect, but the area and perpendicular of the small polygon are found in the Table to the side 1. Hence the reason of the foregoing rules.

The tabular numbers are found by Trigonometry, thus, for the pentagon: divide 360 degrees by 5, and the quotient is 72 degrees for the angle *a o b*, its half, 36 degrees, is the angle *a o c*. Then,

TABLE II.

When the side of the Polygon is 1.

No. of Sides.	Radius of inscribed Circle.	Area of Polygon.	
3	0·2886751	0·4330127 =	$\frac{3}{4}$ tan. 30° $= \frac{1}{4}\sqrt{3}$
4	0·5000000	1·0000000 =	$\frac{4}{4}$ tan. 45° $= 1 \times 1$
5	0·6881910	1·7204774 =	$\frac{5}{4}$ tan. 54° $= \frac{5}{4}\sqrt{(1 + \frac{2}{5}\sqrt{5})}$
6	0·8660254	2·5980762 =	$\frac{6}{4}$ tan. 60° $= \frac{6}{4}\sqrt{3}$
7	1·0382617	3·6339124 =	$\frac{7}{4}$ tan. 64° $\frac{2}{7}$
8	1·2071068	4·8284271 =	$\frac{8}{4}$ tan. 67° $\frac{1}{2} = 2 \times (1 + \sqrt{2})$
9	1·3737387	6·1818242 =	$\frac{9}{4}$ tan. 70°
10	1·5388418	7·6942088 =	$\frac{10}{4}$ tan. 72° $= \frac{5}{2}\sqrt{(5 + 2\sqrt{5})}$
11	1·7028437	9·3656404 =	$\frac{11}{4}$ tan. 73° $\frac{7}{11}$
12	1·8660254	11·1961524 =	$\frac{12}{4}$ tan. 75° $= 3 \times (2 + \sqrt{3})$

Note. The radius of the circumscribing circle, when the side of the polygon is 1, may be seen in Table 1.

The expressions in the fourth column may be seen in *Trigonometry*, to which the pupil is referred for a full investigation of them. The tangents of the angle O a C in the heptagon, nonagon, and undecagon, are extremely difficult to be found without a table of tangents.

1. The side of a pentagon is 20 yards, and the perpendicular from the centre to the middle of one of the sides is 13·76382; required the area?

Sine angle $aoc = 36°$	9·7692187
is to $ac = \cdot 5\ (= \frac{ab}{2} = \frac{1}{2})$	− 1·6989700
so is sine angle $oac = 54°$	9·9079576
	9·6069276
	9·7692187
to perpendicular $co = \cdot 688191$ ..	− 1·8377089

Hence (·688191 × 5) ÷ 2 = 1·720477 = the area of the pentagon whose side is 1. The rest of the numbers in the Table are calculated in a similar manner. (See *Trigonometry*.) These may be found by other methods, some of which are rather difficult. The perpendiculars are the radii of the inscribed circle.

By Rule I. $20 \times 5 \times 13·76382 \div 2 = 1376·382 \div 2 = 688·196$. *Ans.*

By Rule II. $20 \times 20 \times 1·720477 = 688·19$ the area as before.

2. The side of a hexagon is 14, and the perpendicular from the centre 12·1243556? *Ans.* 509·2229352.

3. The side of an octagon is 5·7, required its area? *Ans.* 156·875596479.

4. The side of a heptagon is 19·38 yards, what is its area? *Ans.* 1356·6.

5. The side of an octagon is 10 feet, what is its area? *Ans.* 482·84271.

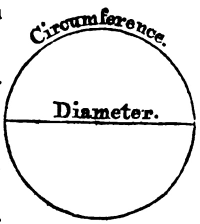
6. The side of a nonagon is 50 inches, what is its area? *Ans.* 15454·5605.

7. The side of an undecagon is 20, what is its area? *Ans.* 3746·25614.

8. The side of a duodecagon is 40 yards, what is its area? *Ans.* 17913·84384.

PROBLEM XVI.

Given the diameter of a circle, to find the circumference; or the circumference to find the diameter, and thence the area.



Rule I. Say as 7 : 22 :: the given diameter : circumference.

Or, as 113 : 355 :: the diameter : the circumference.

Or, as 1 : 3·1416 :: the diameter : the circumference.

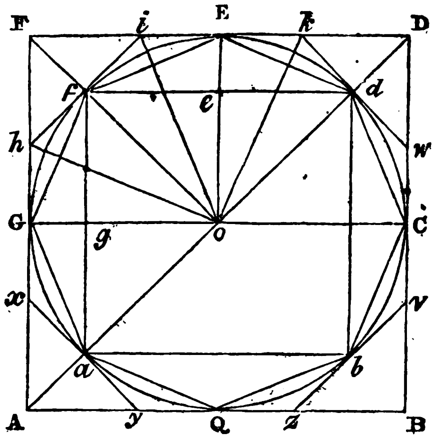
2 — As 22 : 7 :: the given circumference : the diameter.

Or, as 355 : 113 :: the circumference : the diameter.

Or, as 3·1416 : 1 :: the circumference : the diameter.

Demonstration. To establish the truth of these rules: Let A B D F and *a b d f* be two squares, circumscribed about, and inscribed in a circle; let F D be 2, and then the area of A B D F is 4, and of *a b d f* is 2 (7. IV.): thence it is

required to find the areas of two other regular polygons of double the number of sides, (eight,) the one circumscribed, and the other inscribed, viz. $h\,i\,k\,w\,v\,z\,y\,x$, and $G\,f\,E\,d\,C\,b\,Q\,a$.



First, it is evident that $o\,g\,f$ and $o\,G\,F$ are similar, and are also like portions of the squares $a\,b\,d\,f\,a$, and A B D F A; it is also evident that the triangle $o\,G\,f$, and the quadralateral $o\,G\,h\,f$, are like portions of the inscribed and circumscribed polygons, of which the sides $G\,f$ and $G\,h$; as $g\,f$ is parallel to G F, $o\,g : o\,G :: o\,f : o\,F$; but $o\,g : o\,G ::$ the triangle $o\,g\,f : o\,G\,f$ (1. VI.); for the same reason $o\,f : o\,F ::$ the triangle $o\,G\,f :$ the triangle $o\,G\,F$; therefore, by equality of ratios, the triangle $o\,g\,f : o\,G\,f : o\,G\,F$; but these triangles have been shown to be like portions of the inscribed, derivative, and circumscribed polygons; therefore the polygon $G\,f\,E\,d\,C\,b\,Q\,a$ is a mean proportional between $a\,b\,d\,f\,a$ and A B D F A; but $a\,b\,d\,f$ is 2, and A B D F is 4; therefore $G\,f\,E\,d\,C\,b\,Q\,a$ is equal $\sqrt{2 \times 4} = 2{\cdot}8284271$.

Again, because $o\,h$ bisects the angle $G\,o\,F$, $G\,o : o\,F$, or $g\,o : o\,f :: G\,h : h\,F$ (3. VI.); but $g\,o : o\,f :: g\,o : o\,G ::$ the triangle $o\,g\,f : o\,G\,f$; and $o\,g : o\,f\,(= o\,G) :: o\,G :$

o F :: G h : h F :: the triangle o G h : the triangle o h F; then from equality of ratios the triangle $o g f$: the triangle o G f :: o G h : the triangle $o h$ F. Hence the inscribed figure $a b d f$: its derivative inscribed figure G f E d C b Q a : the triangle o G h : the triangle $o h$ F; and as magnitudes are proportionals by conversion, and also by the similar increase of their homologous terms, it follows that $a b d f$ and G f E d C b Q a together are to twice $a b d f$ as the triangles o G h and $o h$ F together (= o G F) to twice the triangle o G h (= o G $h f$); that is, $a b d f$ together with G f E d C b Q a is to A B D F as o G F to o G $h f$; but o G F and o G $h f$ are like portions of A B D F and $x h i k w v z y$, therefore $a b d f$ and G f E d C b Q a together are to A B D F as A B D F to $x h i k w v z y$. But $a b d f = 2$, and G f E d C b Q a = 2·8284271; hence 2 + 2·8284271 : 4 :: 4 (= A B D F) : 3·3137085 = the area $x h i k w v z y$.

From this it appears that the two inscribed polygons are to twice the simple inscribed polygon, as the area of the circumscribing polygon, to the area of the derivative circumscribing polygon having double the number of sides. Hence to find the area of the inscribed polygon of 16 sides, which is a mean proportional between 2·8284271 and 3·3137085, we multiply them together and extract the square root of the product, thus, $\sqrt{(2{\cdot}8284271 \times 3{\cdot}3137085)} = 3{\cdot}0614674$; and to find the area of the circumscribing polygon of 16 sides, we say, as $2{\cdot}8284271 + 3{\cdot}0614674 : 2 \times 2{\cdot}8284271$, or, $5{\cdot}8898945 : 5{\cdot}6568542 :: 3{\cdot}3137085 : 3{\cdot}1825979$.

Pursuing this mode of calculation, namely, extracting the square root, and finding a fourth proportional alternately, the following table may be formed, showing the numbers expressing the areas of the inscribed and circumscribed polygons, which appear from the table to approach each other continually, and therefore to the area of the inscribed circle. Now if we conceive the circumscribing polygon to consist of an infinite number of sides, the sum of all is evidently equal to the circumference of the circle, but the area of such a polygon is found by multiplying half the sum of all the sides by the radius; therefore the area of a circle is found by multiplying half the circumference by the radius; but as the

area, to the radius 1, is 3·1415926, or 3·1416 nearly, it is evident that 3·1416 is half the circumference of a circle whose radius is 1, and the circumferences of circles being as their radii, 3·1416 is the circumference of a circle whose diameter is 1. Hence, if D denote the diameter of any circle, its circumference will be found by the analogy, as 1 : 3·1416 : : D : the circumference; the converse of this is evident.

To find smaller numbers expressing the approximate ratio of 1 to 3·1416, such as 113 to 355, 7 to 22, &c. See GREGORY's *Philosophy of Arithmetic—Continued Fractions.*

And to find the area, having the diameter, by means of a tabular number, it will be $D^2 \times \cdot 78539815$. Because the areas are to each other as the squares of their diameters, it will be, as $2^2 : D^2 : 3{\cdot}1416 :$ the area of the circle whose diameter is D, this $= \frac{D^2}{2} \times 3{\cdot}1416 = D^2 \times \cdot 78539815$.

TABLE III.

When the number of sides is

Number of sides.	Areas of the inscribed Polygon.	Area of the circumscribing Polygon.
4 ..	2·0000000	4·0000000
8 ..	2·8284271	3·3137085
16 ..	3·0614674	3·1825979
32 ..	3·1214451	3·1517249
64 ..	3·1365485	3·1441184
128 ..	3·1403311	3·1422236
256 ..	3·1412772	3·1417504
1024 ..	3·1414729	3·1416025
2048 ..	3·1415877	3·1415951
4096 ..	3·1415914	3·1415933
8192 ..	3·1415923	3·1415928
16384 ..	3·1415925	3.1415927
32768 ..	3·1415926	3·1415926

By finding the area of the inscribed and circumscribed polygons of 393216 side, to the diameter 1, they would be found to agree up to the tenth decimal place, the circumference of the intermediate circle being greater than 3·1415926535, but less than 3·1415926537. Hence the area of a circle is always less than the area of any regular polygon circumscribed about it, and its circumference less than the circumference of the polygon. But the area of a circle is greater than the area of any inscribed polygon, and its circumference greater than the circumference of the polygon, it being the ultimate limit of both. Its area is the greatest within the same perimeter. The diameter and circumference of a circle are incommensurable, and though Geometry furnishes no method of finding a line equal to the circumference, yet the approximate numerical solution given above answers all the purposes to which it may be applied.

From what has been said, it is easy to conceive that the area of a cirele is equal to the area of a right angled triangle, whose altitude is equal to the radius of the circle, and base equal to its circumference.

For more on this subject, see *Trigonometry*.

1. The diameter of a circle is 15, what is its circumference?

$7 : 22 :: 15 : 22 \times 15 \div 7 = 330 \div 7 = 47{\cdot}142857$.

Or, $113 : 355 :: 15 : 355 \times 15 \div 113 = 5325 \div 113 = 47{\cdot}124$.

Or, $1 : 3{\cdot}1416 :: 15 : 3{\cdot}1416 \times 15 = 47{\cdot}124$.

2. The circumference of a circle is 80, what is its diameter?

$22 : 7 :: 80 : 7 \times 80 \div 22 = 25{\cdot}45$.

$355 : 113 :: 80 : 113 \times 80 \div 355 = 25{\cdot}4647$.

$3{\cdot}1416 : 1 :: 80 : 80 \div 3{\cdot}1416 = 25{\cdot}457$.

3. What is the circumference of a circle, whose diameter is 10? *Ans.* 31·4285.

4. What is the diameter of a circle, whose circumference is 50? *Ans.* 15·909.

5. The diameter of the earth is 7958 miles, what is its circumference? *Ans.* 2500·8528 miles.

6. The circumference of the earth being 25000·8528 miles, what is its diameter? *Ans.* 7958 miles.

PROBLEM XVII.

To find the length of the arc of a Circle.

RULE I. Multiply the radius of the circle by the number of degrees in the given arc, and that product by 0·1745329, and the last product will be the length of the arc.*

---

* It has been shown that, when the radius is 1, the semi-circumference is 3·1416, which being divided by 180, the degrees in a semi-circle, gives ·01745329, the length of 1 degree, to the radius 1.

The rest of the rule is evident.

The following theorem taken from Dr. Hutton, is more accurate:

Let $d =$ the diameter, and v the versed sine, then $\left(5\,d\sqrt{\frac{5\,v}{5\,d-3\,v}} + 4\sqrt{d\,v}\right)\times\frac{2}{9} =$ the length of the arc.

Demonstration. From Trigonometry we have $A = S + \frac{S^3}{2\cdot3r^2} + \frac{S^5}{2\cdot4\cdot5\cdot r^4} + \frac{3\cdot5\cdot S^7}{2\cdot4\cdot6\cdot7\cdot r^6}$, &c., that is, $A = S \times \left(1 \times \frac{S^2}{2\cdot3\cdot r^2} + \frac{3\,S^4}{2\cdot4\cdot5\,r^4} + \frac{3\cdot5\cdot S^6}{2\cdot4\cdot6\cdot7\cdot r^6}, \text{ \&c.}\right)$

The length of the curve A D B being required, call its half D B, A, the versed sine D P, v, sine B P, S, and radius r; and because K P $\times$ P D $=$ B P^2; that is, $(2\,r - v)\times v = S^2$, and then $\sqrt{(2\,r\,v - v^2)} = S$; substituting this value of S in the above equation, we get $A = \sqrt{(2\,r\,v - v^2)} \times 1 + \frac{2\,r\,v - v^2}{2\cdot3\cdot r^2} + \frac{3\times(2\,r\,v - v^2)^2}{2\cdot4\cdot5\cdot r^4} + \frac{3\cdot5\times(2\,r\,v - v^2)^3}{2\cdot4\cdot6\cdot7\cdot r^6}$, &c.

By expanding and multiplying these factors, we get

$A = \sqrt{2\,r\,v} \times \left(1 + \frac{v}{2\cdot3\cdot2\,r} + \frac{3\,v^2}{2\cdot4\cdot5\cdot2^2 r^2} + \frac{3\cdot5\cdot v^3}{2\cdot4\cdot6\cdot7\cdot2^3 r^3}, \text{ \&c.}\right)$ Put $2\,r = d$, and we get

$$A = \sqrt{d\,v} \times \left(1 + \frac{v}{2\cdot3\,d} + \frac{3\,v^2}{2\cdot4\cdot5\cdot d^2} + \frac{3\cdot5\,v^3}{2\cdot4\cdot6\cdot7\,d^3}, \text{ \&c.}\right)$$

To find the value of this series, let it be assumed $= d\sqrt{\frac{v}{g\,d - h\,v}} + n\sqrt{d\,v}$, and we get

RULE II. From eight times the chord of half the arc, subtract the chord of the whole arc, and it will be found

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$$A = d\sqrt{\frac{v}{gd-hv}} + n\sqrt{dv} = \sqrt{\frac{d^2 v}{gd-hv}} + n\sqrt{dv} = \sqrt{dv} \times\left(\sqrt{\frac{d}{gd-hv}} + n\right) = \sqrt{dv} \times \left(\sqrt{\left(\frac{1}{g-\frac{hv}{d}}\right)} + n\right) = \sqrt{dv}$$

$$\times\left(\frac{1}{(\sqrt{g-\frac{hv}{d}})} + n\right) = \sqrt{dv} \times \frac{1}{\left(g-\frac{hv}{d}\right)^{\frac{1}{2}}} + n = \sqrt{dv} \times\left(\left[g-\frac{hv}{d}\right]^{-\frac{1}{2}} + n\right).$$

This being expanded, we get

$$\sqrt{dv} \times\left(\frac{1}{g^{\frac{1}{2}}} + \frac{hv}{2dg^{\frac{3}{2}}} + \frac{3h^2v^2}{8d^2g^{\frac{5}{2}}}, \text{\&c.} + n\right)$$

$$= \sqrt{dv} \times\left(\frac{1}{g^{\frac{1}{2}}} + n + \frac{hv}{2dg^{\frac{3}{2}}} + \frac{3h^2v^2}{8d^2g^{\frac{5}{2}}}, \text{\&c.}\right)$$

By comparing with the equation

$A = \sqrt{dv} \times\left(1 + \frac{v}{2\cdot3d} + \frac{3v^2}{2\cdot4\cdot5d^2} + \frac{3\cdot5v^3}{2\cdot4\cdot6\cdot7d^3}, \text{\&c.}\right)$, we get the co-efficients of the like powers of v equal to each other; that is, $\frac{1}{g^{\frac{1}{2}}} + n = 1$; then $n = 1 - \frac{1}{g^{\frac{1}{2}}} = \frac{g^{\frac{1}{2}}-1}{g^{\frac{1}{2}}}$, and $\frac{1}{6} = \frac{h}{2g^{\frac{3}{2}}}$, and $\frac{1}{3} = \frac{h}{g^{\frac{3}{2}}}$: hence $\frac{g^{\frac{3}{2}}}{3} = h$, and $\frac{g^3}{9} = h^2$. Again, we have $\frac{3}{40} = \frac{3h^2}{8g^{\frac{5}{2}}}$; hence; $\frac{1}{5} = \frac{h^2}{g^{\frac{5}{2}}}$; and $\frac{g^{\frac{5}{2}}}{5} = h^2$; therefore $\frac{g^{\frac{5}{2}}}{5} = \frac{g^3}{9}$; hence $g^3 = \frac{9}{5} \times g^{\frac{5}{2}}$; that is, $g^{\frac{6}{2}} = \frac{9}{5} \times g^{\frac{5}{2}}$, and dividing by $g^{\frac{5}{2}}$, we get $g^{\frac{1}{2}} = \frac{9}{5}$, and $g = \frac{81}{25}$; but $h^2 = \frac{g^{\frac{5}{2}}}{5} = \frac{\left(\frac{81}{25}\right)^{\frac{5}{2}}}{5} = \left[\frac{81}{25}\right]^2 \times \frac{9}{25}$, and substituting these in the equation $A = d\sqrt{\frac{v}{gd-hv}} + n\sqrt{dv}$, we get

that one-third of the remainder will give the length of the arc, nearly.*

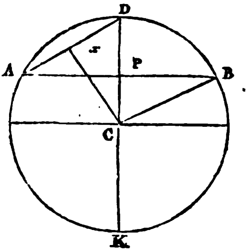
$A = D\sqrt{\left(\frac{v}{\frac{81}{25}d - \frac{81}{25}\times\frac{3}{5}v}\right)} + \frac{4}{9}\sqrt{dv}$: (for $n = \frac{g^{\frac{1}{2}}-1}{g^{\frac{1}{2}}} = \frac{\sqrt{\left(\frac{81}{25}\right)}-1}{\sqrt{\left(\frac{81}{25}\right)}} = \frac{4}{9}) = d\sqrt{\frac{v}{(d-\frac{3}{5}v)\times\frac{81}{25}}} + \frac{4}{9}\sqrt{(dv)} = \frac{\frac{d}{9}}{\frac{}{5}}\sqrt{\left(\frac{v}{d-\frac{3}{5}v}\right)}$ $+\frac{4}{9}\sqrt{dv} = \frac{5d}{9}\sqrt{\frac{v}{d-\frac{3}{5}v}} + \frac{4}{9}\sqrt{dv} = (5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv})\times$, $\frac{1}{9}$; and $2A = (5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv})\times\frac{2}{9}$, which is the rule.

* *Demonstration.* Let the radius C D $= r$, and sine A P $=$ S; then the chord A D $= \sqrt{(S^2 + [r - \sqrt{(r^2 - S^2)}]^2)} = S + \frac{S^3}{8r^2} + \frac{7S^5}{128r^4} +$ &c.; and 8 A D $= 8S + \frac{S^3}{r^2} + \frac{7S^5}{16r^4} +$ &c.; therefore 8 A D $-$ 2 A P $= 6S + \frac{S^3}{r^2} + \frac{7S^5}{16r^4}$, &c.; and $\frac{1}{3}$(8 A D $-$ 2 A P) $= \frac{1}{3}$(8AD $-$ A B) $= 2S + \frac{S^3}{3r^2} + \frac{7S^5}{48r^4}$, &c. But from Trigonometry, the length of the arc A D, whose sine is S, is known to be S $+ \frac{S^3}{6r^2} + \frac{3S^5}{40r^4}$, &c.; and therefore the arc A D B will be $2S + \frac{S^3}{3r^2} + \frac{6S^5}{40r^4}$, &c. Now when we compare this expression for the length of the arc with $\frac{1}{3}$(8 A D $-$ A B) $= 2S + \frac{S^3}{3r^2} + \frac{7S^5}{48r^4}$, &c., we find the difference to be $\frac{S^5}{240r^4}$, which proves the truth of the rule.

The reason of the rule may be otherwise shown thus: Let A B D $= 2$ A, A D $= c$, A P $=$ S; then by Trigonometry, A $=$ S $+ \frac{S^3}{2\cdot3r^2}$ + &c., and 2 A $= 2S + \frac{S^3}{3\cdot r^2}$. Also, $\frac{A}{2} = \frac{c}{2} + \frac{\left(\frac{c}{2}\right)^3}{2\cdot3r^2}$, &c.; then

1. If the arc A B contain 30 degrees, the radius being 9 feet, what is the length of the arc?

Ans. $30 \times 9 = 270$, and $270 \times \cdot 01745329 = 4 \cdot 7124$.



2. If the chord A D of half the arc A D B be 20 feet, and the chord A B of the whole arc 38, what is the length of the arc?

Ans. $20 \times 8 - 38 = 122$; then $122 \div 3 = 40\frac{2}{3}$ feet, nearly.

3. The chord of an arc is 6 feet, and the chord of half the arc is $3\frac{1}{2}$, required the length of the whole arc?

Ans. $7\frac{1}{3}$.

4. The chord of the whole arc is 40, and the versed sine 15; what is the length of the arc? *Ans.* $53\frac{1}{3}$.

5. The chord A B is 18, and the versed sine D P 4; what is the diameter? *Ans.* $24\frac{1}{4}$.

6. The chord A B of the whole arc is 48·74, and the chord A D of half the arc 30·25; required the length of the arc? *Ans.* 64·42.

7. $A B = 30$, $D P = 8$; required the length of the arc? *Ans.* $35\frac{1}{3}$.

$A = C + \frac{c^3}{8 \cdot 3\, r^2}$; and $8\,A = 8\,c + \frac{c^3}{3\,r^2}$, &c.; hence $8\,A - 2\,A = 6\,A = 8\,c + \frac{c^3}{3\,r^2}$, &c., $- 2\,S - \frac{S^3}{3\,r^2}$, &c., but $\frac{c^3}{3\,r^2}$ being nearly equal to $\frac{S^3}{3\,r^2}$, therefore $6\,A = 8\,c - 2\,S$, and $2\,A\ (= A\,D\,B) = \frac{8\,c - 2\,S}{3}$ which is the rule.

PROBLEM XXVIII.

To find the area of a Circle.

RULE I. Multiply half the circumference by half the diameter, for the area.*

RULE II. Multiply the square of the diameter by ·7854, for the area.†

RULE III. Multiply the square of the circumference by ·07958.‡

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* This rule is demonstrated in Problem XVI.

† The reason of this rule may be seen in the note to Problem XVI.

‡ When the circumference of a circle is 1, its diameter is ·318309, and its area is $\frac{\cdot318309}{2} \times \frac{1}{2}$ (note to Problem XVI.) $= \frac{\cdot318309}{4} =$ ·079577 = ·07958 nearly; but circles are to each other as the squares of their circumferences; therefore $1^2 : \text{cir.}^2 :: \cdot07958 : \text{area} = \text{cir.}^2 \times \cdot07958$, cir. being any circumference.

$7 : 22 ::$ diameter $1 :$ circumference $= \frac{22}{7}$, and the area is $\frac{22}{7\times2} \times \frac{1}{2} = \frac{11}{14}$; hence $14 : 11 :: d^2 :$ area $5\cdot22 : 7 ::$ circumference $1 :$ diameter $= \frac{7}{22}$, and $\frac{7}{22\times2} \times \frac{1}{2} = \frac{7}{88}$ the area of a circle whose circumference is 1 : therefore $1^2 : \text{cir.}^2 :: \frac{7}{88} :$ area; that is $88 : 7 : \text{cir.}^2 :$ area cir. being the circumference.

Cor. Hence if $d =$ diameter, $c =$ circumference, $a =$ area, and $n = 3\cdot1416$. Then

1st. $d = \frac{c}{n} = \frac{4a}{c} = 2\sqrt{\left(\frac{a}{n}\right)}$. For $n : 1 :: c : d = \frac{c}{n}$, and $\frac{d}{2} \times \frac{c}{2} = a$; then $cd = 4a$ and $d = \frac{4a}{c}$, again $1 : n :: d : nd = c$; then $\frac{4a}{c} = \frac{4a}{nd} = d$; then $d^2 = \frac{4a}{n}$, and $d = \sqrt{\left(\frac{4a}{n}\right)} = 2\sqrt{\left(\frac{a}{n}\right)}$.

2nd. $c = nd = \frac{4a}{d} = 2\sqrt{(na)}$. For $cd = 4a \because c = \frac{4a}{d}$, again $c = \frac{4a}{d} = \frac{4a}{\frac{c}{n}} = \frac{4an}{c} \therefore c^2 = 4an$ and $c = 2\sqrt{(an)}$.

RULE IV. As 14 to 11, so is the square of the diameter to the area.

RULE V. As 88 to 7, so is the square of the circumference to the area.

1. To find the area of a circle whose diameter is 100, and circumference 314·16.

By RULE I.	By RULE II.	By RULE III.
3·1416	·7854	98696·5 sq. cir.
100	$100^2 =$ 10000	·07958
4)31416	Area 7854	7854· Area.
Area 7854		

By RULE IV.	By RULE V.
$100^2 =$ 10000	98696·5 sq. cir,
11	7
2)110000	8)690875·5
7)55000	11)86359·4
Area 7857	7850·85

2. What is the area of a circle whose diameter is 7 ? *Ans.* $38\frac{1}{2}$.

3. How many square yards are in a circle whose diameter is $1\frac{1}{6}$ yard ? *Ans.* 1·069.

3rd. $a = \frac{nd^2}{4} = \frac{c^2}{4n} = \frac{dc}{4}$. For $c = nd \because \frac{d}{2} \times \frac{nd}{2} = a = \frac{nd^2}{4}$, and $c^2 = 4an \therefore a = \frac{c^2}{4n}$; likewise $c = \frac{4a}{d} \therefore cd = 4a$, hence $a = \frac{cd}{4}$.

4th. $n = \frac{c}{d} = \frac{4a}{d^2} = \frac{c^2}{4d}$. For $c = nd \because n = \frac{c}{d}$ and $a = \frac{nd^2}{4}$; then $4a = nd^2 \because n = \frac{4a}{d^2}$; again, $d^2 = \frac{4a}{n} \because nd^2 = 4a$ and $n = \frac{4a}{d^2}$.

4. The surveying wheel turns twice in the length of $16\frac{1}{2}$ feet; in going round a circular bowling-green it turns exactly 200 times; how many acres, roods, and perches in it?
Ans. 4 acres 3 roods 35·8 perches

5. The circumference of a fish-pond is 56 chains, what is its area? *Ans.* 249·56288.

6. What is the area of a quadrant, the radius being 100?
Ans. 7854.

7. Required the length of a cord fastened to a stake at one end and to a cows horns at the other, so as to allow her to feed on an acre of grass, and no more?
Ans. $39\frac{1}{4}$ yards.

8. The circumference of a circle is 91, what is its area?
Ans. 650·00198.

9. The diameter of a circle is 15 perches, what is its area? *Ans.* 176·715.

10. What is the area of the semicircle of which 20 is the radius? *Ans.* 628·32

PROBLEM XIX.

Given the diameter of a circle, to find the area of a square equal in area to the circle.

RULE. Multiply the diameter by ·8862269, and the product will be the side of a square equal in area.*

1. If the diameter of a circle be 100, what is the side of a square equal in area to the circle? *Ans.* 88·62269.

2. The diameter of a circular fish-pond is 200 feet, what is the side of a square fish-pond equal in area to the circular one? *Ans.* 177·24538.

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* *Demonstration.* When the diameter is 1, the area is ·7854, (Prob. XVI.); then the side of a square equal in area to ·7854 is $\sqrt{(\cdot7854)} = \cdot8862269$, the multiplier.

PROBLEM XX.

Given the circumference of a circle, to find the side of a square equal in area to the circle.

RULE. Multiply the circumference by ·2820948, and the product will be the side of the square.*

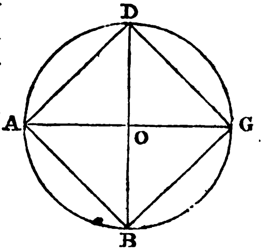
1. The circumference of a circle is 100, what is the side of a square equal in area to the circle? *Ans.* 28·20948.

2. The circumference of a round fish-pond is 200 yards, what is the side of a square fish-pond equal in area to the round one? *Ans.* 56·41896.

PROBLEM XXI.

Given the diameter to find the side of the inscribed square.

RULE. Multiply the diameter by ·7071068, and the product will give the side of the inscribed square.†



1. The diameter of a circle is 100, what is the side of the inscribed square? *Ans.* 70·71068.

2. The diameter of a circle is 200, what is the side of the inscribed square? *Ans.* 141·42136.

---

* *Demonstration.* When the circumference is 1, the area of the circle is ·079577, (Prob. XVIII. Rule 3.); then the side of the square equal in area to ·079577 is $\sqrt{(\cdot 079577)} = \cdot 2820948$, the multiplier.

Note. When the area of the circle is given, the side of the square is found by extracting the square root of the area.

† *Demonstration.* When the diameter is 1, (= B D), the area of the circumscribed square is 1, and therefore the area of the inscribed square is $\frac{1}{2}$ (= ·5), and the side itself is $\sqrt{\cdot 5} = \cdot 7071068$.

Note. When the diameter A C is 1, the side E H is 1.

PROBLEM XXII.

Given the area of a circle, to find the side of the inscribed square.

RULE. Multiply the area by ·6366197, and extract the square root of the product which will give the side of the inscribed square.*

1. The area of a circle is 100, what is the side of the inscribed square? *Ans.* 7·97884.

2. The area of a circle is 200, what is the side of the inscribed square?

$200 \times \cdot 6366197 = 127 \cdot 3239400$; then $\sqrt{127 \cdot 3239400} = 11 \cdot 2837$. *Ans.*

PROBLEM XXIII.

Given the side of a square, to find the diameter of a circumscribing circle.

RULE. Multiply the side of the square by 1·4142136, and the product will give the diameter of the circumscribing circle.†

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* *Demonstration.* Let the area of the circle A B C D be 1; then from the first equation in the Note to Prob. XVIII. $d = 2\sqrt{\left(\frac{1}{3 \cdot 1416}\right)}$, and $\frac{d}{2} = \sqrt{\left(\frac{1}{3 \cdot 1416}\right)}$; hence $A\,o = \sqrt{\left(\frac{1}{3 \cdot 1416}\right)}$, and $A\,o^2 = \frac{1}{3 \cdot 1416}$; but $A\,o^2 + o\,D^2 = 2\,A\,o^2 = (47 \cdot 1) = A\,D^2 = \frac{2}{3 \cdot 1416} = \cdot 6366197$. Now similar figures are as the squares of their like sides ∴ 1 : given area :: ·6366197 ($= A\,D^2$) to the square of the side of the inscribed square = given area × ·6366197 ∴ the side itself is ($\sqrt{}$(given area × ·6366197),) which is the rule.

† *Demonstration.* When the side of the square is 1, the radius of the circumscribing circle is ·7071068, and the diameter is $\cdot 7071068 \times 2 = 1 \cdot 4142136$. See Table I.

1. If the side of a square be 10, what is the diameter of the circumscribing circle? *Ans.* 14·142136.

If the side of a square be 20, find the diameter of the circumscribing circle? *Ans.* 28·284272.

PROBLEM XXIV.

Given the side of a square, to find the circumference of the circumscribing circle.

RULE. Multiply the side of the square by 4·4428934, and the product will be the circumscribing circle.*

1. If the side of a square be 100, what is the circumference of the circumscribing circle? *Ans.* 444·28934.

2. If the side of the square be 30, what is the circumference of the circumscribing circle? *Ans.* 133·286802.

PROBLEM XXV.

Given the side of a square, to find the diameter of a circle equal in area to the square.

RULE. Multiply the side of the square by 1·128791, and the product will be the diameter of a circle equal in area to the square whose side is given.†

1. If the side of a square be 100, what is the diameter of the circle whose area is equal to the square whose side is 100? *Ans.* 112·83791.

2. What is the diameter of a circle equal in area to a square, whose side is 200? *Ans.* 225·67582.

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* When the side of the square is 1, the diameter of the circumscribing circle is 1·4142136; and therefore its circumference 1·4142136 $\times$ 3·1416 $=$ 4·4428934 : hence the reason of the rule.

† *Demonstration.* From the first equation in the note to Problem XVIII., we have $d = \sqrt{\left(\frac{4a}{n}\right)}$; but when the side of a square is 1, its area is 1 $\therefore d = \sqrt{\left(\frac{4a}{n}\right)} = \sqrt{\left(\frac{4}{3{\cdot}1416}\right)} = 1{\cdot}1283791$, &c. Hence the reason of the rule.

PROBLEM XXVI.

Given the side of a square, to find the circumference of a circle, whose area is equal to the square whose side is given.

Rule. Multiply the side of the square by 3·5449076, and the product will give the circumference of a circle equal in area to the given square.*

1. What is the circumference of a circle, whose area may be equal to a square whose side is 100? *Ans.* 354·49076.

2. Find the circumference of a circle equal in area to a square whose side is 300? *Ans.* 1063·47228.

PROBLEM XXVII.

To find the area of a sector of a circle.

Rule I. Multiply half the length of the arc by the radius of the circle, and the product is the area of the sector.†

Rule II. As 360 is to the degrees in the arc of the sector, so is the area of the whole circle to the area of the sector.‡

1. Let A C B O be a sector less than a semi-circle, whose

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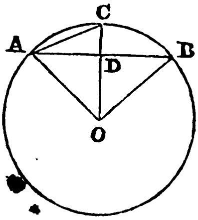
* *Demonstration.* By the last Problem, the diameter of a circle equal in area to a square, whose side is 1, is 1·1283791, and its circumference therefore will be 1·1283791 × 3·1416 = 3·5449076.

† The reason of these operations is evident from Problems XVII. and XVIII.

‡ The reason of the second rule is evident from the 33. VI. which proves that arcs of the same circle are as the sectors: therefore the whole circumference (360) is to the given arc (18) as the area of the whole circle is to the area of the sector.

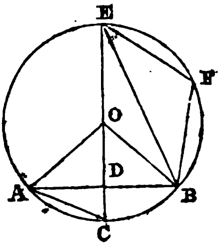


radius A O is 20 feet, and chord A B 30 feet; what is the area?

Ans. First, $\sqrt{(AO^2 - AD^2)} = \sqrt{(400 - 225)} = 13{\cdot}228 = OD$; then $OC - OD = 20 - 13{\cdot}228 = 6{\cdot}772 = CD$.

Again, $\sqrt{(AD^2 + CD^2)} = \sqrt{(225 + 45{\cdot}859984)} = 16{\cdot}4578 = AC$ the chord of half the arc.

Hence, by Problem XVII. the arc A B is 37·2208; then $\frac{37{\cdot}2208}{2} \times 20 = 372{\cdot}208$ the area required.



2. Let A E F B O A be a sector greater than a semi-circle, whose radius A O is 20, the chord E B 38, and chord B F of half E F B 23; required the area?

$$\begin{array}{rl} 23 & = \text{chord BF} \\ 8 & \\ \hline 184 & \\ 38 & = \text{chord BE} \\ \hline 3)146 & \\ \hline 48{\cdot}666, \&c. & = \text{arc BFE} \\ 20 & \\ \hline 973\tfrac{1}{2} & \text{area.} \end{array}$$

3. What is the area of a sector, whose arc contains 18 degrees, the diameter being 3 feet?

$$\begin{array}{r} 7854 \\ 9 \\ \hline \end{array}$$

Then 360 : 18 :: 7·0686 : the area of the sector;
Or, 20 : 1 :: 7·0686 : ·35343. *Ans.*

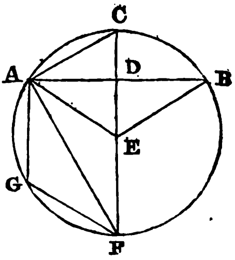
4. What is the area of a sector whose arc contains 147 degrees 29 minutes, and radius 25? *Ans.* 804·3986.

5. What is the area of a sector, whose arc contains 18 degrees, the radius being 3 feet? *Ans.* 1·41371849.

PROBLEM XXVIII.

To find the area of the segment of a circle.

RULE I. Find the area of the sector having the same arc with the segment, by the last Problem; find also the area of the triangle, formed by the chord of the segment and the two radii of the sector. Then add these two areas together, when the segment is greater than a semi-circle, but find their difference when it is less than a semi circle, for the answer.*



1. What is the area of the segment A C B D A, its chord A B being 24, and radii A E or E C 20 $\sqrt{(AE^2 - AD^2)} = \sqrt{(400 - 144)} = 16 = DE$, $EC - ED = 20 - 16 = 4 = CD$, $\sqrt{(AD^2 + DC^2)} = \sqrt{(144 + 256)} = 20 = AC$; then $\frac{(20 \times 8) - 24}{3} = 45\frac{1}{3}$ = arc A C B,

And $22\frac{2}{3}$	= half arc	12	= A D
20	= radius	16	

$453\frac{1}{3}$ = area of sector E B C A — 192 = area of $\triangle$ A B E

∴ 192 = area of $\triangle$ A B E

$261\frac{1}{3}$ = area of segment A B C A.

2. Let A G F B A be a segment greater than a semicircle; there are given the chord A B 20·5, F D 17·17, A F 20, T G 11·5, and A E 11·64, required the area of the segment?

$\frac{(FG \times 8) - AF}{3} = \frac{(11{\cdot}5 \times 8) - 20}{3} = 24$ the length

* This rule is evident from the example, and from Problems IV. and XXVII. When the triangle E A B is deducted from the sector *E B C A*, the segment A C B remains,

of the arc A G F (Problem XVII.); then $24 \times 11{\cdot}64 = 279{\cdot}36$, area of sector A E B F G A (Problem XXVII.) Again, F D — E D $= 17{\cdot}17 - 11{\cdot}64 = 5{\cdot}53 =$ E D; then, $\frac{AB \times ED}{2} = \frac{20{\cdot}5 \times 5{\cdot}53}{2} = 56{\cdot}6825$ the area of the triangle A B E, which being added to the area of the sector before found will give the area of the segment, viz. $279{\cdot}36 + 56{\cdot}6825 = 336{\cdot}0425$ the area of the segment A G F B A.

RULE II. To two-thirds of the product of the chord and versed sine of the segment, add the cube of the versed sine divided by twice the chord, and the sum will give the area of the segment nearly.

When the segment is greater than a semi-circle, find the area of the remaining segment, and deduct it from the area of the whole circle, the remainder will give the area of the segment.*

---

* *Demonstration.* This rule, which is the best for practice, was originally demonstrated by Mr. Peter Nicholson. He however took the fundamental part of his demonstration from Dr. Hutton, which the Doctor discovered by Fluxions. Let $v =$ versed sine C D of the arc A C B, and d the diameter of the circle; then by a formula in the Trigonometry, (which see), the length of half the arc is $\sqrt{(dv)} \times \left(1 + \frac{v}{6d} + \frac{3v^2}{40d^2}, \text{ \&c.}\right)$ which multiplied by half the diameter will, by Prob. XXVII., give the area of the sector, that is, the area of the sector is $\frac{d}{2}\sqrt{(dv)} \times \left(1 + \frac{v}{6d} + \frac{3v^2}{40d^2}, \text{ \&c.}\right)$ Now it is easy to conceive that $\pm \frac{1}{2} d \mp v =$ the altitude of the triangle whose base A B is $2\sqrt{(dv - v^2)} = 2\sqrt{(dv)} \times \left(1 - \frac{v}{2d} - \frac{v^2}{8d^2}, \text{\&c.}\right)$ Hence the area of the triangle is $(\pm \frac{1}{2} d \mp v) \times \sqrt{(dv)} \times \left(1 - \frac{v}{2d} - \frac{v^2}{8d^2}, \text{ \&c.}\right)$, which being added to, or subtracted from, the area of the sector, gives $2v\sqrt{(dv)} \times \left(\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2}, \text{ \&c.}\right)$ But D F $\times$ D C $=$ A D^2 (35, III.); that is, $(d - v) \times v = \frac{c^2}{4}$, ($c$ being put for A B); therefore $d = \frac{c^2}{4v} + v$, and $2v\sqrt{(dv)} = v\sqrt{(c^2 + 4v^2)}$

3. What is the area of the segment A C B, less than a semi-circle, its chord being 18·9, and height or versed sine D C 2·4. ?

Ans. $AB \times DC = 18{\cdot}9 \times 2{\cdot}4 = 45{\cdot}36$, and $\frac{2}{3} AB \times DC = \frac{2}{3} \times 45{\cdot}36 = 30{\cdot}20666$; then $\frac{2{\cdot}4^3}{2 \times 18{\cdot}9} = {\cdot}36571$; hence $30{\cdot}20666 + {\cdot}36571 = 30{\cdot}57237$ the area.

4. Required the area of the segment A G F B whose height F D is 20, and chord A B 20 ?

$\frac{AB}{2} = \frac{20}{2} = 10 = AD$, and $AD^2 = 100$; but $AD^2 = FD \times DC \therefore CD = \frac{AD^2}{FD} = \frac{100}{20} = 5$.

The area of the segment A C B is, by the last case, 69·7916; and the area of the whole circle, by Prob. XVIII. is 490·87; then $490{\cdot}87 - 69{\cdot}7916 = 421{\cdot}0834$ = area of the segment A G F B.

5. What is the area of the segment A G F B, greater than a semi-circle, whose chord A B is 12, and versed sine 18 ? *Ans.* 297·81034

RULE III. 1. Divide the height of the segment by the diameter of the circle, to three places of decimals.

2. Find the quotient in the column height, and take out the corresponding area segment, which multiply by the square

---

$= vc + \frac{2v^3}{c} - \frac{2v^5}{c^3} + \frac{4v^7}{c^5}$, &c., (by extracting the square root of $c^2 + 4v^2$, and multiplying by v); therefore, $2v\sqrt{}(dv \times \left(\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2}\right) \doteq \left(vc + \frac{2v^3}{c}\right) \times \left(\frac{2}{3} - \frac{v}{5d}\right) = \frac{2}{3} vc \times \frac{4v^3}{3c} - \frac{vc^2}{5d}$, &c., $= \frac{2}{3} vc + \frac{4v^3}{3c} - \frac{4vc^3}{5c^2 + 20v^2} = \frac{2}{3} vc + \frac{8c^2v^3 + 80v^5}{15c^5 + 60vc^9} = \frac{2}{3} vc + \frac{8v^3}{15c} - \frac{32v^4 + 80v^5}{15c^3 + 60vc^2} = \frac{2}{3} vc + \frac{8v^3}{15c}$ nearly, which is the rule, See Problem XVII.

of the diameter, and the product will be the area of the segment required.*

Note I. If the quotient of the height by the diameter be greater

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* *Demonstration.* To prove the truth of this rule, it will be necessary to show that segments whose versed sines are as the diameters, will be to each other as the squares of the diameters.

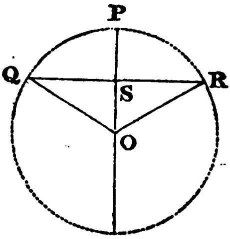
Let A E B A and *a e b a* be two similar segments, cut from the similar sectors A E B C A and *a e b c a*, by the chords A B and *a b*. Draw C E bisecting both the arcs.

By similar triangles C A : C *a* :: C S : C *s*; that is, C A : C *a* : C A − E S : C E − *e s* ∴ C A : C *a* :: E S : *e s*. Hence the versed sines of similar segments are as the radii of the circles, or as the diameters; but similar sectors arcs as the squares of the diameters, and similar triangles as the squares of their like sides; C A^2 : C a^2 :: sector A E B C A : sector *a e b* C *a* :: triangle C A B : triangle C *a b* :: seg. A E B A (= sector A E B C A − triangle A B C) : seg. *a e b a* (= sector *a e b* C *a* − triangle *a b* C); that is, the segments are to each other as the squares of the diameters.

Now, the diameter in the tables is 1, then by putting $d =$ any diameter, and $v =$ versed sine, we shall have $d : v :: 1 : v \div d =$ the versed sine of a similar segment in the table, whose area we shall call a. Then from what has been proved $1^2 : d^2 :: a : a d^2 =$ area of the segment, whose height, or versed sine is v, and diameter d. The table of seg. is at the end of work.

Note. If to the square of half the chord of the segment there be added the square of the versed sine, the square root of the sum will give the chord of half the arc of the segment. To $\frac{4}{3}$ of the chord of half the arc of the segment add the chord of the segment, the sum multiplied by $\frac{2}{5}$ of the versed sine will give the area. The truth of this rule may be shown thus:—

As in Rule II., we have $A = \sqrt{(dv)} \times \left(1 + \frac{v}{2 \cdot 3\, d} + \frac{3 v^2}{2 \cdot 4 \cdot 5 d^2}, \text{ \&c.}\right)$



and therefore the area of the sector Q P R O is $\frac{d}{2} \times \left(\sqrt{(dv)} \times (1 +\right.$

than ·5 subtract it from 1, and find the area segment corresponding to the remainder, which subtract from ·7854 for the correct area segment.

Note II. If the quotient of the height by the diameter does not terminate in three figures, find the area segment corresponding to the first three decimal figures of the quotient, subtract it from the next greater area segment, multiply the remainder by the fractional part of the quotient, and add the product to the area segment first taken out of the table. When great accuracy is not required, the fractional part may be omitted.

$\frac{v}{2 \cdot 3\, d} + \frac{3\, v^2}{2 \cdot 4 \cdot 5\, d^2})\Big)$ but Q S $\times$ S O $=$ area of the triangle Q O R; that is, $\sqrt{((d - v) \times v)} \times \left(\frac{d}{2} - v\right) = \sqrt{(d\, v - v^2)} \times \left(\frac{d}{2} - v\right) =$ area of the triangle Q O R; and this expanded, and taken from the area of the sector as found before, we get $2\, v \times \left(\sqrt{(d\, v)} \times \frac{2}{3} - \frac{v}{5\, d} - \frac{v^2}{28 d^2}, \text{\&c.}\right)$ for the area of the segment in terms of v and d, which assume equal to $2\, v\, (m \sqrt{(d\, v - v^2)} + n \sqrt{(d\, v)})$ (in order to find a finite value for the segment) $= 2\, v\, (m \sqrt{(d\, v} \times \left(1 - \frac{v}{d}\right)) + n \sqrt{(d\, v)}) =$ $2\, v\, (m \sqrt{(d\, v} \times \left(1 - \frac{v}{d}\right)^{\frac{1}{2}} + n \sqrt{(d\, v)}) = 2\, v\, (\sqrt{(d\, v)} \times (m \times \left(1 - \frac{v}{d}\right)^{\frac{1}{2}}$ $+ n) = 2\, v \sqrt{(d\, v)} \times [m \times \left(1 - \frac{v}{2 d} - \frac{v^2}{8 d^2}\right) + n] = 2\, v \times \sqrt{(d\, v)}$ $\times \left(m - \frac{m\, v}{2 d} - \frac{m\, v^2}{8 d^2}, \text{\&c.} + n\right) = 2\, v \times \sqrt{(d v)} \times m + n - \frac{m\, v}{2 d}$ $- \frac{m\, v^2}{8 d^2}$, &c. Now, by comparing these two expressions for the area of the segment, we get $m + n = \frac{2}{3}, - \frac{m}{2} = - \frac{1}{5}$; therefore $m = \frac{2}{5}$; then $\frac{2}{5} + n = \frac{2}{3}$, and $n = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$. By substituting these in the equation expressing the segment, we get

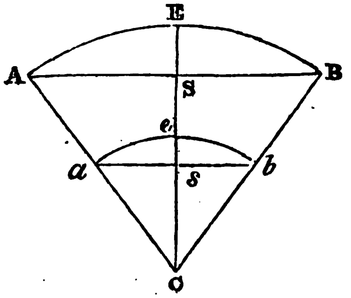
$$2\, v \times \frac{2}{5} \sqrt{(d\, v - v^2)} + \frac{4}{15} \sqrt{(d\, v)} = \frac{2}{5}\, v \times (2 \sqrt{(d\, v - v^2)} + \frac{4}{3} \sqrt{(d\, v)};$$

but $\sqrt{(d\, v)}$ is the chord of half the arc, and $2 \sqrt{(d\, v - v^2)}$ is the chord of the segment, and $\frac{2}{5}\, v$ is $\frac{2}{5}$ of the versed sine; therefore $\frac{2}{5}\, v \times \left(2 \sqrt{(d\, v - v^2)} + \frac{4}{3} \sqrt{(d\, v)}\right)$ expresses the rule.

6. Let the diameter be 20, and the versed sine 2, required the area of the circle?

$\frac{2}{20} = \cdot 1$, to which answers $\cdot 040875$
Square of diameter, 400

16·35 area.



7. What is the area of a segment, whose diameter is 52, and versed sine 2?

$\frac{2}{52} = \cdot 0384\frac{8}{13}$ which is the tabular versed sine. Then to ·0384 answers ·009917, and the difference between this area and the next is ·000038, which multiplied by $\frac{8}{13}$ gives ·000023, which added to ·009917 gives ·009940, which is the area corresponding to the versed sine $\cdot 0384\frac{8}{13}$. Then $52^2 \times \cdot 009940 = 26 \cdot 878787$ is the area required.

PROBLEM XXIX.

To find the area of a Zone, or the space included by two parallel chords and the arcs contained between them.

RULE. Join the extremities of the parallel chords towards the same parts, and these connecting lines will cut off two equal segments, the areas of which added to the area of the trapezoid will give the area of the zone.*

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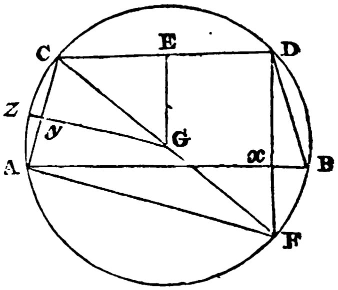
* *Demonstration.* The trapezoid A B D C, together with the segments A Z C A and D B F D is equal to the zone between the chords



1. Suppose the greater chord $AB = 30$, the less CD 20, and the perpendicular distance $Dx = 25$, required the area of the zone $ABDC$. $\frac{1}{2}(AB - CD) = xB = \frac{1}{2}(30-20) = 5$: then $\sqrt{(Dx^2+xB^2)} = DB = \sqrt{(25^2+5^2)} = 25{\cdot}49$. $AB - Bx = Ax = 30 - 5 = 25$, and $(Ax \times Bx) \div Dx = Fx = (25 \times 5) \div 25 = 5$. $Dx + Fx = DF = 25 + 5 = 30 = \frac{1}{2}\sqrt{(CD^2 + DF^2)} = \frac{1}{2} CF = Gz = \frac{1}{2}\sqrt{(20^2+30^2)} = 18{\cdot}025$, the radius of the circle; $(DB \times Ax) \div 2Dx = Gy = (25{\cdot}49 \times 25) \div (2\times25) = 12{\cdot}745$, $Gz - Gy = zy$; $18{\cdot}025 - 12{\cdot}745 = 5{\cdot}28$ the height of the segment AzC. $36{\cdot}05)5{\cdot}28({\cdot}146$ the tabular area segment answering to which is $\cdot071033$, then $\cdot071033 \times (36.05)^2 = 92{\cdot}315 =$ the area of the segment $Az\,C$.

CD and AB; but the area of the segments is found by Problem XXVIII., and the area of the trapezoid is found by Problem XIII.

Draw AB of the given length, and make xB equal to half the difference between the chords AB, CD; then raise the perpendicular xD equal to the distance between the given chords, draw DC parallel to AB, and join AC. Bisect AC, CD in the points y, E, from which erect the perpendiculars yG, EG, and where they meet at G will be the centre of the circle; from the centre G with the radius GC describe a circle; produce CG to F, and CF will be the diameter of the circle. Produce Dx which will meet the diameter at F; because EG is parallel to DF, and bisects both CD and CF; join AF, and CAF is a right angle. As Gy bisects AC and CF, it is parallel to AF, and is also equal to half of AF. Because $\frac{1}{2}(AB - CD) = xB$, and $\sqrt{(Dx^2 + xB^2)} = DB$; $AB - xB = Ax$; then $Ax \times xB = Dx \times xF$ $\therefore$ $xF = Ax \times xB \div Dx$; but $Dx + xF = DF$, and $\sqrt{(CD^2 + DF^2)} = CF$, the half of which is $CG = GF$.

Again, the triangles AxF and DxB are similar; $Dx : DB :: Ax : AF$ which is double Gy; then $Gy = DB \times Ax \div 2Dx$.

Lastly, $Gz - Gy = yz$ the height of the segment AzC.

When the diameter of the circle is given, or the chord AC and height yz, the operation is very simple, in which case Dx need not *be given.*

$\frac{1}{2}$ (A B + C D) × D x = $\frac{1}{2}$ (30 + 20) × 25 = 625 the area of the trapezoid A B D C. Hence 625 + 92·315 × 2 = 809·63 = the area of the zone.

2. Let the chord A B = 48, the chord C D = 30; the chord A C = 15·8114; what is the area of the zone A B D C?

Ans. The diameter C F = 50, height of the segment A z C = 1·2829, area by the table of segments = 13·595. Area of the zone A B D C = 534·19.

3. Let A B = 20, C D = 15, and their distance 17$\frac{1}{2}$; required the area? *Ans.* 395·4369.

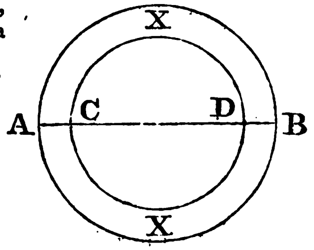
4. Let A B = 96, C D = 60, and their distance 26; required the area? *Ans.* 2136·7527.

PROBLEM XXX.

To find the area of a circular ring, or of the space included between two concentric circles.

Rule. Multiply the sum of the two diameters by their difference and the product arising by ·7854 for the area of the ring.*

1. The diameter A B is 30, and C D 20; what is the area of the ring X X?



30
20

50 sum
10 difference

500
·7854

392·7000 area of the ring X X.

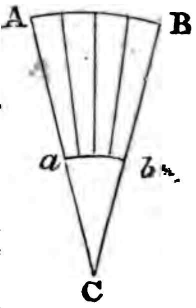
* Let D be the diameter of the larger circle, and d the diameter of the smaller; then $D^2 \times \cdot 7854$ = area of the larger circle, and $d^2 \times \cdot 7854$ = area of the smaller circle ∴ $D^2 \times \cdot 7854 - d^2 \times \cdot 7854 = (D^2 - d^2) \times \cdot 7854 = (D + d) \times (D - d) \times \cdot 7854$ = the area of the ring. And this expression corresponds with the rule.

2. What is the area of the circular ring, when the diameters are 40 and 30? *Ans.* 549·78.

3. What is the area of a circular ring, when the diameters are 50 and 45? *Ans.* 373·065.

PROBLEM XXXI.

To find the area of a part of a ring, or of the segment of a sector.



RULE. Multiply half the sum of the bounding arcs by their distance asunder, and the product will give the area.*

1. Let A B be 50, and $a\,b$ 30; and the distance a A 10; what is the area of the space $a\,b$ B A?

$$Ans.\ \frac{50+30}{2} \times 10 = 400.$$

2. Let A B $= 60$, $a\,b = 40$, and aA $= 2$; required the area of the space $a\,b$ B A? *Ans.* 100.

3. Let A B $= 25$, $a\,b = 15$, and a A $= 6$, required the area of the segment of the sector? *Ans.* 120.

---

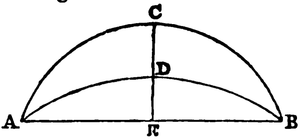
* Let A B $=$ A and $a\,b = a$; let, also, A C the radius $= r$. Then, as similar arcs are to each other as their radii, A B : $a\,b$ :: C A : Ca; that is, A : a :: r : $\frac{a\,r}{A} = a$C; but A C $-$ a C $= a$A $= r - \frac{a\,r}{A} = \frac{A\,r - a\,r}{A} = \frac{(A-a)}{A} \times r$. But the area of the sector A C B $=$ $\frac{A\,r}{2}$, and the area of the sector a C $b = \frac{C\,a \times a\,b}{2} = \frac{a\,r}{A} \times \frac{a}{2} = \frac{a^2\,r}{2\,A}$

$\therefore$ the area of the segment A B $b\,a = \frac{A\,r}{2} - \frac{a^2\,r}{2\,A} = \frac{A^2\,r - a^2\,r}{2\,A} = \frac{(A^2 - a^2)}{2\,A} \times r = \frac{(A+a)}{2} \times \frac{(A-a)}{A} \times r = \frac{A+a}{2} \times A\,a$, by substituting A a for its equal $\frac{A-a}{A} \times r$, which is the rule.

PROBLEM XXXII.

To find the area of a Lune, or the space included between the intersecting arcs of two eccentric circles.

RULE. Find the areas of both segments which form the lune, and deduct the less from the greater for the answer.*



1. Let the chord A B = 40, E C = 12, and E D = 4; what is the area of the lune A D B C A?

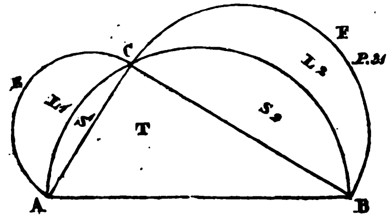
Ans. By the 35. III. $(AE^2 \div EC) + EC =$ diameter of the circle of which A C B is an arc, and $(AE^2 \div ED) + ED =$ the diameter of the circle of which A D B is an arc; hence $(20^2 \div 12) + 12 = 45{\cdot}3'$; and $(20^2 \div 4) + 4 = 104$; are the two diameters.

$12 \div 45{\cdot}3 = {\cdot}264$. $4 \div 104 = {\cdot}038$.

The area segment answering to ·264 is ·165780, and $(45{\cdot}3)^2 \times {\cdot}165780 = 340{\cdot}1954802 =$ area of the segment A E B C A.

The area segment answering to ·038 is ·009763, and $(104)^2 \times {\cdot}009763 = 105{\cdot}596608 =$ area of the segment A E B D A; then $340{\cdot}1954802 - 105{\cdot}596608 = 234{\cdot}5988722 =$ the area of the lune.

2. Let the chord A B = 40, and the heights of the segments E C and E D are 15 and 2; required the area of the lune? *Ans.* 388·47384.



* *Demonstration.* It is self-evident that the segment A C B A — A D B A = the line A C B D A.

If A B C be a right angled triangle, on the three sides of which, if three semi-circles be described; then the triangle T (A B C) will be equal to the sum of the two lunes L 1, L 2. Because the sum of the semi-circles described on the sides containing the right angle is equal to the semi-circle described on the hypothenuse, and taking away the segments S 1, S 2, which are common to the equal quantities, the remainders will be equal, viz. the sum of the lunes L 1, L 2, will remain equal to T.

PROBLEM XXXIII.

TO MEASURE LONG IRREGULAR FIGURES.

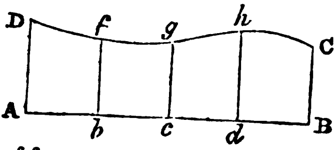
When irregular figures, not reducible to any known figure, present themselves, their contents are best found by the method of equi-distant ordinates.

RULE. Take the breadth in several places, at equal distances, and divide the sum of the first and last ordinate by 2 for the arithmetical mean between those two. Add together this mean and all the other breadths, omitting the first and last, and divide their sum by the number of parts so added, the quotient will give the mean breadth of the whole, which being multiplied by the given length will give the area of the figure, very nearly.

It is not necessary sometimes to take the breadths at equal distances, but to compute each trapezoid separately, and the sum of all the separate areas thus found will give the area of the entire, nearly.

Or, add all the breadths together and divide by the number of them for a mean breadth, which being multiplied by the length, as before, will give the area nearly.

1. Let the ordinate A D be 9·2, *b f* 7, *c g* 9, *d h* 10, B C 8·8, and the length A B 30; required the area?



9·2 A D
8·8 B C

2)18

9 mean breadth of first and last.
7 *b f*
9 *c g*
10 *d h*

4)35 sum

8·75 mean breadth of all.
30

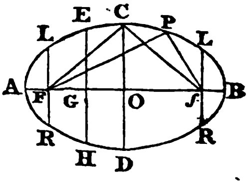
262·50 area of the whole figure.

CONIC SECTIONS.

SECTION III.

OF THE ELLIPSES.

1. If two pins be fixed at the points F*f*, and a thread P*f*F P be put round them and knotted at P; then if the point P and the thread be moved about the fixed points F*f*, so as to keep the thread always stretched, the point P will describe the curve A C P B D A called an ellipsis.



Let the pupil fix two pins in a table at any convenient distance, as at F*f*; next he is to fasten the two ends of a thread, and throw it loosely over the fixed pins; then by stretching the string with a black lead pencil, or a sharp pointed instrument, and carrying it gently round, an ellipsis will be formed.

2. The two points or centres F*f* where the pins are fixed, are called the foci.

3. The line, passing through the foci, is called the transverse axis, or the axis major. The point O, in the middle of the axis A B, is the centre of the ellipsis.

4. The line C D, drawn through the centre of the ellipsis, perpendicular to the transverse axis A B, is called the conjugate axis, or the axis minor.

5. The lines L R, drawn through the focal points F*f*, perpendicular to the transverse axis A B, is called the parameter, or latus rectum.

6. A line drawn from any point of the curve, perpendicular to the transverse axis, is called an ordinate to the transverse axis, as E G, or H G. When it goes quite through the ellipsis, as E H, it is called a double ordinate.

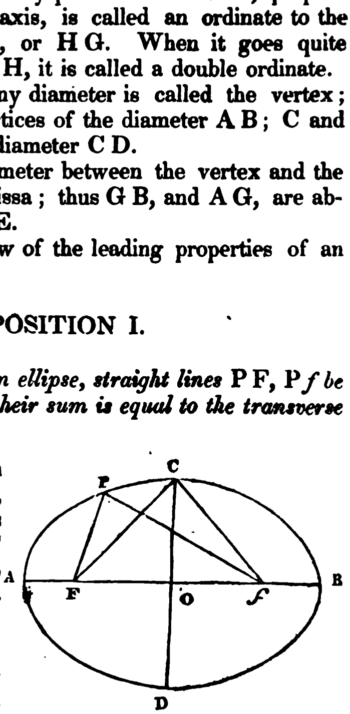
7. The extremity of any diameter is called the vertex; thus, A and B are the vertices of the diameter A B; C and D are the vertices of the diameter C D.

8. That part of the diameter between the vertex and the ordinate is called an abscissa; thus G B, and A G, are abscissas to the ordinate G E.

The following are a few of the leading properties of an ellipsis:

PROPOSITION I.

If from any point P, *in an ellipse, straight lines* P F, P f *be drawn to the foci* F f, *their sum is equal to the transverse axis* A B.



Demonstration. From the generation of the curve, it is evident that Af is equal to B F; hence A F $= Bf$. It is plain also that $FP + Pf = Af + AF = Af + Bf = AB$.

Cor. Hence it appears that if two right lines be drawn from every point in the curve to the foci, the sum of every two connecting lines is the same, and equal to the transverse axis.

PROPOSITION II.

The line connecting the extremity of the conjugate axis and focus of the ellipse is equal to half the transverse axis; that is, FC *or* $fC = \frac{AB}{2} = AO$, *or* OB.

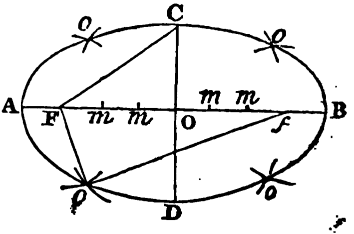
Demonstration. By the last Proposition, $FC + fC =$

A B; but F C $=f$ C; because in the triangles F o C and fo C, F o is equal to fo and o C common, and the angles at o right; therefore F C is equal to fC (4. I.) and hence F C is equal to $\frac{AB}{2}$ = A O, or OB.

PROPOSITION III.

THEOREM.

The transverse and conjugate diameters of an ellipsis being given, to find the foci, and construct the figure.



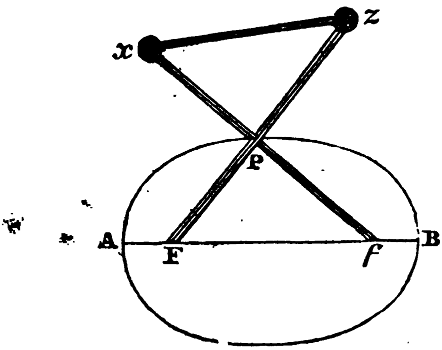
Draw the transverse and conjugate diameters, bisecting each other at right angles at O; from C as a centre and radius A O, describe an arc cutting the transverse diameter A B in F f, which are the foci of the ellipsis; take a great number of points in A B, the more the better, as m, m, &c. with the radii A m, B m, (A m + B m = A B) and with F f as centres, describe two arcs crossing each other at o o, &c. join o o, &c. with the hand, and the curve will be that of an ellipsis.

Demonstration. By Proposition II. F C, $=f$C = A O: and by Proposition I., F o + of = A m + B m = A B; hence the reason of the construction.

On the same principle, an ellipse may be constructed by means of three rulers.

Provide three rulers, of which two F z, fx are equal, each, to the transverse axis A B, and the third z x equal to the focal distance F f. Then connecting these rulers so as to move freely About F f, and also about x, z, their inter-

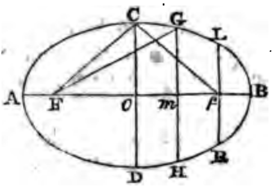
section P will always be in the curve; so that if slits run along both rulers, and the instrument turned freely about the foci, a pencil, or sharp-pointed instrument introduced through the slits at the point of intersection, will describe the ellipse.



PROPOSITION IV.

The distance between the two foci is a mean proportional between the sum and difference of the transverse and conjugate axis, that is, $AB+CD : Ff :: Ff : AB-CD$.

Demonstration. $CO^2 = FC^2 - FO^2 = AO^2 - FO^2$ (Prop. II.) $\therefore 4CO^2 (= CD^2) = 4AO^2 - 4FO^2$; but $4AO^2 = AB^2$, and $4FO^2 = Ff^2$ (Cor. 4. II.); hence $Ff^2 = AB^2 - CD^2 = (AB+CD) \times (AB-CD)$ (Cor. 5. II.); therefore $AB+CD : Ff :: Ff : BA - CD$ (17. VI.)



PROPOSITION V.

The square of the distance of the focus from the centre, is equal to the difference of the squares of the semi-axes; that is, $FO^2 = AO^2 - CO^2$.

Demonstration. $FO^2 = FC^2 - OC^2$ (47. I.); but $FC = AO$ (Prop. II.); therefore $FO^2 = AO^2 - OC^2$. —See the last figure.

PROPOSITION VI.

The rectangle of the distance of the focus from each vertex, is equal to the square of the semi-conjugate; that is, $AF \times FB = CO^2$.

Demonstration. Because $CO^2 = FC^2 - FO^2$; but $FC = AO$; therefore $CO^2 = AO^2 - FO^2 = AF \times FB$, (Cor. 5. II.) See the last figure.

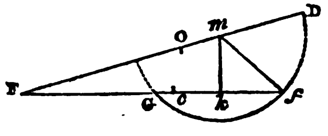
PROPOSITION VII.

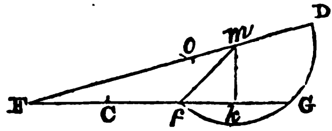
The square of half the transverse axis is to the rectangle of the greatest and least focal distance from the extremities of the transverse, as the rectangle of the abscessas to the square of the ordinate which divides them; that is, $Ac^2 : AF \times FB :: Ak \times kB\, mk^2$.

To understand this, it is necessary to premise the following lemma, viz.

If from the vertex of any plane triangle a perpendicular be let fall on the base, or base produced, and the base bisected; then half the base is to half the sum of the sides, as the difference between half the sum of the sides and one of them to the distance between the middle of the base and perpendicular.

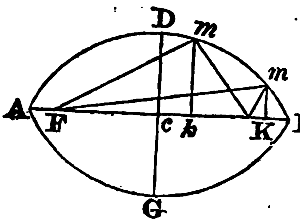
Make $FO =$ half the sum of the sides Fm, fm; then it will be, $Fc : FO :: Om$, or $FO - fm : CK$.





Demonstration. Because $Ff : FD :: Fm - fm : FG$. Hence $Fc : FO :: Om : \frac{1}{2}FG = \frac{1}{2}Ff \pm \frac{1}{2}fG = fC \pm fK = CK$.

By the lemma $AC : FC :: Ck : AC - fm$, which being squared and divided, will be $AC^2 : AC^2 - FC^2 :: CK^2 : CK^2 - AC^2 + 2AC \times fm - fm^2$, which being involved and divided, will be $AC^2 : AC^2 - FC^2 :: AC^2 - CK^2 : 2AC^2 - FC^2 - CK^2 - 2AC \times fm + fm^2$;



but $AC^2 - FC^2 = FD^2 = AF \times FB$ (Prop. VI.); also $AC^2 - CK^2 = Ak \times kB$, and $2AC^2 - 2AC \times fm = 2FC \times CK$; likewise $fm^2 - FC^2 - CK^2 + 2FC \times Ck = fm^2 - fk^2 = mk^2$ $\therefore AC^2 : AF \times FB :: Ak \times kB : mk^2$.

Cor. 1. From this it follows that, the square of half the transverse is to the square of half the conjugate, as the rectangle of any two abscissas, is to the square of the ordinate which divides them.

For $AF \times FB = CD^2$ (Prop. VI.) $\therefore AC^2 : CD^2 :: Ak \times kB : mk^2$.

Cor. 2. The transverse axis is to the latus rectum, or parameter, as the rectangle of any two abscissas is to the square of the ordinate which divides them. Because $DG^2 = AB \times p$, (putting p for the parameter); therefore $AB^2 : AB \times p :: Ak \times kB : mk^2$; hence $AB : p :: Ak \times kB : mk^2$.

Cor. 3. Hence the rectangles of every pair of abscissas are proportional to the squares of their corresponding ordinates.

Cor. 4. The square of the conjugate is to the square of the transverse, as the rectangle of any two abscissas is to the square of the corresponding ordinates.

For $AC^2 : CD^2 :: AC^2 - Ck^2 : mk^2$ (Cor. 2.), or CN^2; then by inversion and division, $CD^2 : AC^2 :: CD^2 - CN^2 (= DN \times NG) : Ck^2 = (mN^2)$.

Cor. 5. The rectangle of the focal distances is to the square of half the transverse, as the rectangle of any two abscissas of the conjugate, to the square of the ordinate which divides them.

For $AF \times FB = CD^2$ (Prop. VI.) $\therefore AF \times FB : AC^2 :: DN \times NG : Nm^2$.

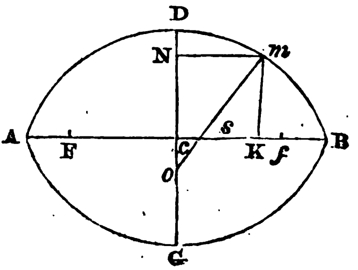
Cor. 6. Hence the conjugate is to its parameter, as the rectangles of the abscissas, to the square of the ordinates which divides them.

Cor. 7. Hence also, the rectangles of the abscissas of the conjugate are proportional to the squares of the ordinates which divide them.

Cor. 8 As $AC^2 (= FD^2) = CD^2 + FC^2$; then $CD^2 + FC^2 : CD^2 :: Ak \times kB : mk^2$.

Cor. 9. By *Cor.* 4. $CD^2 : CD^2 + FC^2 :: DN \times NG : Nm^2$.

Cor. 10. As $CD^2 = AC^2 - FC^2$; then, by Cor. 4. $AC^2 - FC^2 : AC^2 :: DN \times NG : Nm^2$.



Cor. 11. Because $AB = \frac{DG^2}{p}$; therefore by Cor. 2, $DG^2 : p^2 :: AK \times kB : mk^2$.

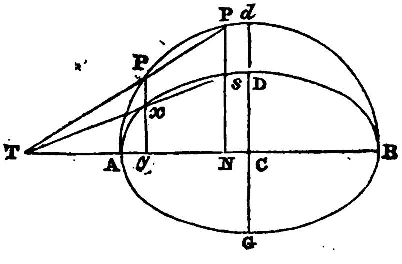
Cor. 12. As $DG = \frac{AB^2}{p}$; then by Cor. 6, $AB^2 : p^2 :: DN \times NG : Nm^2$.

Cor. 13. From m draw $mo =$ half the transverse, then will $ms =$ half the conjugate. For by similar triangles mo^2 or $AC^2 : Ck^2 :: sm^2 : sk^2$; then by inversion and division $AC^2 : sm^2 :: AC^2 - Ck^2 : (sm^2 - sk^2 =)$ $mk^2 \because sm = CD$.

Cor. 14. Therefore if from the point o be laid off $os =$ half the difference between the diameters, and that line produced till it becomes equal to half the transverse, its extremity will be in the curve.

Cor. 15. If upon either axis a circle be described, the corresponding circular and elliptic ordinates will be proportional. Because the rectangles of the abscissas are equal to the squares of the circular ordinates. (35. III.)

Cor. 16. If through the extremities of two unequal elliptic ordinates a straight line be drawn, so as to cut the axis produced, a straight line drawn from the point of contact will

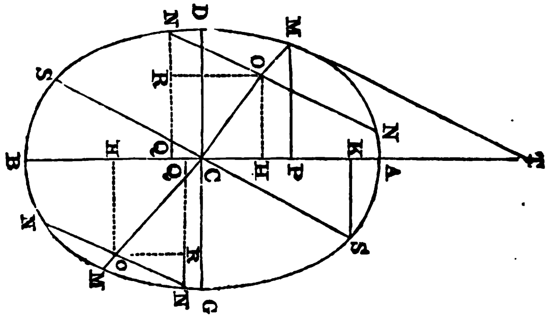


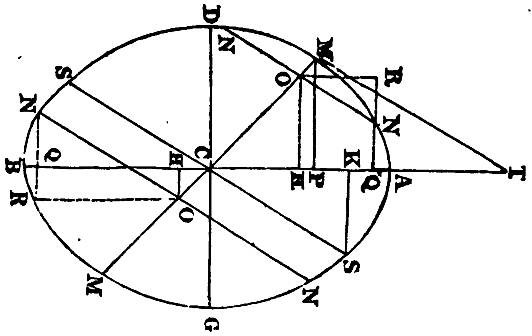
pass through the extremities of the corresponding circular ordinates. Because $PN : py :: SN : xy$ (Cor. 15.); but $SN : xy :: TN : Ty$ $\therefore$ $PN : py :: TN : Ty$.

Cor. 17. From this we may infer that tangents drawn to any two corresponding ordinates will pass through the same point in the axis.

PROPOSITION VIII.

In the ellipse, the square of any diameter is to the square of its conjugate, as the rectangle of the abscissas to that diameter, to the square of the ordinate which divides them; that is, $CM^2 : CS^2 :: MO \times OM : ON^2$.





From the point O draw O R parallel to the axis A B, and O H perpendicular to it: also through the point N, the extremity of the ordinate, draw R Q parallel to D G, and draw the ordinates M P and K S.

Then by similar triangles $NQ = \left(\frac{CH}{CP} \mp \frac{OR}{TP}\right) \times MP$;

that is, $(CP \times PT)^2 : MP^2 :: CH^2 \times TP^2 \mp 2CH \times OR \times CP \times PT + OR^2 \times CP^2 : NQ^2$, but $AP \times PB : MP^2 :: AC^2 - CH^2 \mp 2CH \times OR - OR^2 : NQ^2$ (Prop. VII. Cor. 3.);* and $AP \times PB = CP \times PT$; therefore $CH^2 \times TP^2 \mp 2CH \times OR \times CP \times PT + OR^2 \times CP^2 = (AC^2 - CH^2 \mp 2CH \times OR - OR^2) \times CP \times PT$; hence OR^2

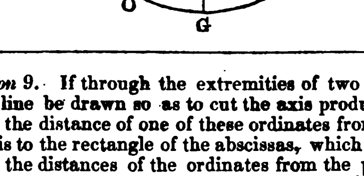
$$= \frac{AC^2 \times CP \times PT - CH^2 \times (CP \times PT + TP^2)}{CP \times PT + CP^2}$$

but $AC^2 = TC \times CP = CP \times PT + CP^2$; and $CP \times PT + TP^2 = CT \times TP$; therefore

$$= \frac{PT \times TC \times CP^2 - PT \times TC \times CH^2}{TC \times CP} = OR^2;$$

that is, $CP : PT :: CP^2 - CH^2 : OR^2$; but by similar triangles $CS^2 : CK^2 : NO^2 : OR^2$; that is, $CP \times PT : CS^2 :: OR^2 : ON^2$ (for $CK^2 = AP \times PB = CP \times PT$); hence by composition $CP^2 : CS^2 :: CP^2 - CH^2 : ON^2$; but by similar triangles and division, $CP^2 : CH^2 :: CM^2 : CO^2$, and $CP^2 : CM^2 :: CP^2 - CH^2 : CM^2 - CO^2$; therefore Ex. Equo. $CM^2 : CS^2 :: CM^2 - CO^2 (= MO \times OM) : NO^2$.

From the property of the curve and similar triangles TQ^2



* *Proposition* 9. If through the extremities of two unequal ordinates a right line be drawn so as to cut the axis produced; then as the square of the distance of one of these ordinates from the point of intersection, is to the rectangle of the abscissas, which it divides, so is the sum of the distances of the ordinates from the point of intersection, to the sum or difference of the distances of the ordinates from the centre according as they fall on the same or contrary sides of the centre; that is, $TQ^2 : AQ \times QB : TQ + TL : CQ \pm CL$.

$: AQ \times QB :: TL^2 : AL \times LB$, and by division $TQ^2 : AQ \times QB :: (TL^2 - TQ^2) : (AL \times LB - AQ \times QB ::) \; 2TC \times CQ \mp 2TC \times CL + CL^2 - CQ^2 : CQ^2 - CL^2$; which divided by $CQ \mp CL$; $TQ : AQ \times QB :: (2TC - CQ \mp CL : CQ \pm CL ::) \; TQ + TL : CQ \pm CL$.

Cor. 1. When Q and L coincide, TM will become a tangent, and CQ, CL will become equal to CN; therefore $TN^2 : AN \times NB :: 2TN : 2CN$; that is, $CN : BN :: AN : TN$.

Cor. 2. By dividing the last analogy, we get $CN : BN - CN :: AN : TN - AN$; that is, $CN : AC :: AN : AT$.

Cor. 3. By inverting and compounding the first analogy, we get $CN : CN + AN :: BN : BN + TN$; that is, $CN : AC :: BN : BT$.

Cor. 4. From the two last corollaries, we get $AN : BN :: AT : BT$.

Cor. 5. By inverting and compounding the second corollary, we get $CN : CN + AN :: AC : AC + AT$; that is, $CN : CA :: CA : CT$.

Cor. 6. By inverting and compounding the last, $CT : CA :: CT + CA : CA + CN$; that is, $CT : CA :: BT : BN$.

Cor. 7. By inverting and dividing the 5th, we get $CT : CA :: CT - CA : CA - CN$; that is, $CT : CA :: AT : NA$.

Cor. 8. By inverting and dividing the 6th, $CT : BT :: CP - AC : BT - BN$; that is, $CT : BT :: AT : NT$.

Cor. 9. By the 6th and 8th, $CA : BN :: AT : NT$.

Cor. 10. By the 7th and 8th, $CA : BT :: AN : NT$.

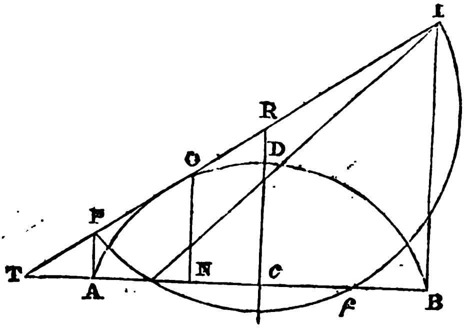
Cor. 11. If tangents be drawn to each vertex of the curve, and if the conjugate be produced to meet any other tangent, then as one of these tangents is to the ordinate drawn from the point of contact, so is the semi-conjugate produced, to the other tangent. For (Cor. 8.) $CT : BT :: AT : NT \because CR : BI :: AP : NO$; hence $BI : NO :: CR : AP$.

Cor. 12. By similar triangles T B : T N :: B I : N O ∵ by the 10th Cor. A C : A N :: B I : O N.

Cor 13. By similar triangles T A : T N :: A P : N O; therefore by the 9th Cor A C : B N :: A P : N O.

Cor. 14. The 12th and 13th Ex. Equo. perturbate A N : B N :: A P : B I.

Cor. 15. By compounding the 12th and 13th A C^2 : A N



$\times$ N B :: A P $\times$ B I : O N^2; that is, C D^2 : N O^2 :: A P $\times$ B I : N O^2 (Prop. I. Cor. 1.); therefore C D^2 = A P $\times$ B I; that is, the semi-conjugate axis is a mean proportion between the vertical tangents.

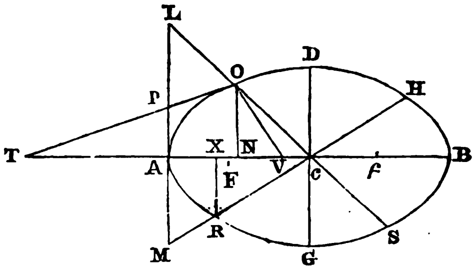
Cor. 16. Hence A P $\times$ B I (= C D^2) A F $\times$ F B = A f $\times$ f B; therefore A F : A P :: B I : F B, and A f : A P :: B I : f B; therefore the triangles are similar and right, and the triangles P N I and P f I are right-angled at F and f; therefore R is the centre of a circle which will pass through the points P, F, f, and I.

PROPOSITION X.

If from two conjugate diameters ordinates be drawn to the axis; the distance from one ordinate to the centre is a mean proportion between the segments of the other; that is, C x^2 = A N $\times$ N B, or C N^2 = A x $\times$ x B.

By similar triangles T N^2 : C x^2 :: [N O^2 : R x^2 ::]

$AN \times NB : AC^2 - Cx^2$ [Prop. I. Cor. 3.] But $AN \times NB = TN \times NC$ [Prop. II. Cor. 1.]; and by Prop. II.



Cor. 5, $AC^2 = TC \times CN$; therefore $TN : Cx^2 :: NC : TC \times CN^2 - Cx^2$, and by composition $TN : TC :: Cx^2 : TC \times CN$; that is, $Cx^2 = TN \times NC = AN \times NB = AC^2 - CN^2$.

Cor. 1. By the 35. III. $AN \times NB =$ the square of the ordinate of the circle applied to the point N; therefore the distance between the centre and one of the ordinates, is equal to the other ordinate produced to the circle.

Cor. 2. Hence [and Prop. I. Cor. 16.] $AC : CD :: Cx : NO$. Likewise $AC : CD :: CN : Rx$; therefore $AC^2 : CD^2 :: NC \times Cx : Rx \times NO$.

Cor. 3. Produce the conjugate diameter till it meets a tangent at the vertex of the curve; then by similar triangles $CN : NO :: AC : AL$; and $Cx : xR :: AC : AM$; therefore $NC \times Cx : Rx \times NO :: AC^2 : LA \times AM$; then by Cor. 2, $CD^2 = LA \times AM$.

Cor. 4. Since by the 2nd $AC^2 : CD^2 :: Cx^2 : NO^2$, and $AC^2 : CD^2 :: CN^2 : Rx^2$; therefore $AC^2 : CD^2 :: Cx^2 + CN^2 : NO^2 + Rx^2$:

But by the Proposition $Cx^2 + CN^2 = AC^2$; therefore $CD^2 = NO^2 + Rx^2$; that is, the square of half the conjugate is equal to the sum of the squares of the ordinates drawn from the conjugate to the axis.

Cor. 5. From this it appears that the sum of the squares of any two conjugate diameters is equal to the sum of the

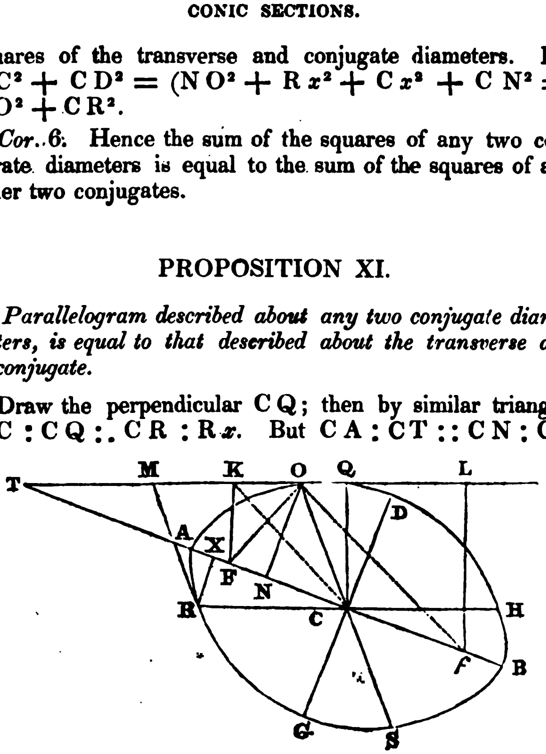
squares of the transverse and conjugate diameters. For $AC^2 + CD^2 = (NO^2 + Rx^2 + Cx^2 + CN^2 =)$ $CO^2 + CR^2$.

Cor. 6. Hence the sum of the squares of any two conjugate diameters is equal to the sum of the squares of any other two conjugates.

PROPOSITION XI.

A Parallelogram described about any two conjugate diameters, is equal to that described about the transverse and conjugate.

Draw the perpendicular CQ; then by similar triangles $TC : CQ :: CR : Rx$. But $CA : CT :: CN : CA$



(Prop. VIII. *Cor.*), and $CN : Rx :: AC : CD$, (Prop. X. Cor. 2.); therefore by composition $CA : CQ :: CR : DC$; hence $AC \times CD = CQ \times CR$.

Cor. 1. Therefore all parallelogram described about the conjugate diameters of an ellipsis are equal.

Cor. 2. By similar triangles CK or $AC : CQ :: fO : fL$; therefore by the Proposition $fO : fL :: CR : CD$; also $FO : FK :: CR : CD$; that is, as the distance of the focus from the point of contact, is to the perpendicular from the focus to the tangent, so is the conjugate diameter parallel to the tangent, to the conjugate axis.

Cor. 3. Hence f L is as $\frac{f\,O}{C\,R}$, and F K as $\frac{F\,O}{C\,R}$.

Cor. 4. By compounding the two last analogies of the 2nd Corollary $F\,O \times O\,f : f\,L \times F\,K :: C\,R^2 : C\,D^2$. But $C\,D^2 = f\,L \times F\,K$; therefore $F\,O \times O\,f = C\,R^2$; that is, the semi-conjugate diameter parallel to any tangent is a mean proportional between the distances of the foci from the point of contact.

PROPOSITION XII.

If upon either axis of the ellipsis a circle be described, the area of the circle will be to that of the ellipsis, as the axis upon which the circle was described, to its conjugate.

By Prop. VII. Cor. 15, the circular is to the corresponding elliptic ordinate, as the axis upon which the circle is described, to its conjugate axis; therefore the sum of all the circular ordinates, or the area of the circle, is to the sum of all the elliptic ordinates, or the area of the ellipsis, as the diameter upon which the circle is described, to the conjugate diameter.

Cor. 1. Therefore the ellipsis is a mean proportional between the circumscribed and inscribed circles.

Cor. 2. Hence also, a circle whose diameter is a mean proportional between the axes is equal to the ellipsis.

Cor. 3. From this Proposition, it appears that all ellipses are as their circumscribing parallelograms.

Cor. 4. Therefore the area of an ellipse is in a ratio compounded of the transverse, and the sub-duplicate of the transverse and parameter, that is, in sub-duplicate ratio of the parameter and the sesquiplicate ratio of the transverse.

Cor. 5. The areas of ellipses, whose transverse diameters are equal, are as the conjugates, or in the sub-duplicate ratio of the parameter.

Cor. 6. The areas of ellipses, whose parameters are equal, are in a sesquiplicate ratio of their greater axes.

Cor. 7. The area of any circular segment is to the area of the corresponding elliptical segment, as the transverse to the conjugate, or generally as the diameter upon which the circle is described, to its conjugate diameter.

PROPOSITION XIII.

The sphere is to the inscribed spheroid as the square of the transverse to the square of the conjugate.*

This follows from the nature of the circle and ellipsis. Because the areas of any two corresponding circles in each, are as the squares of the diameters; that is, as the square of the transverse to the square of the conjugate; therefore the sum of all the sperical circles, is to the sum of all the elliptical circles, as the square of the transverse to the square of the conjugate; but the sum of all the spheric circles is the sphere, and the sum of all the elliptic circles is the spheroid; therefore the sphere is to the spheroid, as the square of the transverse to the square of the conjugate.

Cor. 1. The cylinder circumscribing the sphere is to the circumscribing spheroid, as the square of the transverse to the square of the conjugate; therefore the spheroid is $\frac{2}{3}$ the circumscribing cylinder.

Cor. 2. Let t and c be the transverse and conjugate, and $n = \cdot 7854$; $\frac{2\,c^2\,t\,n}{3}$ is the solidity, or volume of the speroid.

Cor. 3. The corresponding segments of the sphere and spheroid, are as the square of the transverse to the square of the conjugate, and consequently in the same ratio with the solids.

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* A spheroid is a solid, generated by the rotation of a semi-ellipsis about one of its axes, which remains fixed. When the ellipsis revolves about the transverse axis, the figure is called a prolate speroid, which resembles an egg; when the ellipsis revolves about the shorter axis, the figure is called an oblate spheroid, which resembles an orange.

Cor. 4. Hence and from the equation of the curve, the volume of any segment may be found. Let x = any abscissa, and y = the corresponding ordinate; then $y^2 = \frac{c^2}{t^2} \times (t\,x - x^2)$, and therefore the sum of all the circles constituting the volume of the segment, will be (by the Arithmetic of Infinities,) $\frac{4\,n\,c^2}{t^2}\ (\frac{1}{2}\,t\,x^2 - \frac{1}{3}\,x^3) = \frac{2\,n\,c^2}{3\,t^2} \times (3\,t\,x^2 - 2\,x^3)$, or, $\frac{2\,n\,p}{3\,t^2} \times (3\,t\,x^2 - 2\,x^3)$ in terms of the parameter.

Cor. 5. The spheroid is to the inscribed sphere, as the transverse to the conjugate. Because their circumscribing cylinders are as the transverse to the conjugate.

PROPOSITION XIV.

The oblate spheroid is to the inscribed sphere, as the square of the transverse is to the square of the conjugate.

Because the corresponding circles in both solids are as the square of the transverse to the square of the conjugate; therefore the sum of all the corresponding circles must be in that ratio; that is the oblate spheroid is to the inscribed sphere as the square of the transvere is to the square of the conjugate.

Cor. 1. The oblate spheroid is $\frac{2}{3}$ the circumscribing cylinder; therefore the volume of the oblate spheroid is equal to $\frac{2\,n\,t^2\,c}{3}$.

Cor. 2 The corresponding segments of the spheroid and inscribed sphere are as the square of the transverse to the square of the conjugate.

Cor. 3. Therefore the solidity of a segment whose altitude is x, will be
$\frac{4\,n\,t^2}{c^2} \times (\frac{1}{2}\,c\,x^2 - \frac{1}{3}\,x^3) = \frac{2\,n\,t^2}{3\,c^2} \times (3\,c\,x^2 - 2\,x^3)$, or

$\frac{2np}{3t} \times [3cx^2 - 2x^3]$ in terms of the parameter, &c.

Cor. 4. The oblate spheroid is to the circumscribed sphere, as the conjugate to the transverse. Because their circumscribing cylinders are in the ratio of the conjugate to the transverse.

Cor. 6. If about the two axes of an ellipse there be generated two spheres, and two spheroids, the four solids will be continued proportionals; and the common ratio will be that of the two axes of the ellipse; that is, as the sphere upon the greater axis is to the oblate spheroid, so is the oblate spheroid to the prolate spheroid; and as the oblate spheroid is to the prolate spheroid, so is the prolate spheroid to the less sphere, and so is the transverse to the conjugate.

Cor. 7. The oblate spheroid is to the inscribed sphere, as the circumscribed sphere to the prolate spheroid.

Cor. 8. From Cor. 6, the prolate spheroid is a mean proportional between the oblate spheroid and the inscribed sphere.

Cor. 9. And also, the oblate spheroid is a mean proportional between the prolate spheroid and the circumscribed sphere.

Cor. 10. The circumscribed sphere is a mean proportional between the oblate and prolate spheroid.

Note. These equations derived from the general propositions will be shown at full length in a subsequent part of the work.

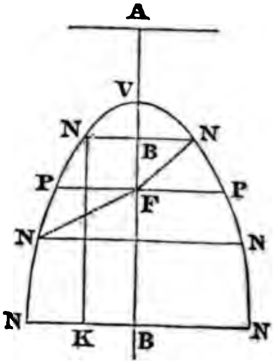
OF THE PARABOLA.

DEFINITIONS.

1. If in an indefinite right line A B, any two points A and F be assumed, and from the centre F, with the radius A B, a circle be described intersecting N N perpendicular to A B, in the points N N; these two points will be in the

curve. If an infinite number of such points be found, the curve passing through them will be a parabola.

2. The point F is called the focus.



3. A right line passing through the points A and F is called the axis.

4. A right line drawn from any point in the curve parallel to the axis, is called a diameter, as N K.

5. A right line drawn from the curve to any diameter, parallel to a tangent at the vertex, is called an ordinate.

6. When an ordinate is perpendicular to its diameter, it is said to be rightly applied, as N B.

7. The distance between the vertex of any diameter, and the intersection of an ordinate, is called the abscissa.

8. The perpendicular P P passing through the focus, is the parameter to the axis A B.

9. A line drawn at right angles to the axis at A is called the directrix.

PROPOSITION I.

The parameter P P, *is four times the focal distance from the vertex.*

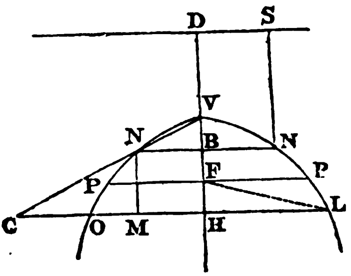
Because F A = F P = 2 F V = 2 V A; therefore P P = 4 F V.

Cor. 1. The vertex V, bisects A F; that is, F V = V A.

PROPOSITION II.

The sum of any abscissa and focal distance is equal to the distance of the focus from the extremity of the ordinate; that is, F N = V B + V F.

Because F N = D B = V B + V F (= N S).



PROPOSITION III.

Every ordinate is a mean proportional between its abscissa and the parameter of the axis; that is, $4\ VF \times BV = BN^2$. *(See the above figure.)*

Because $FN^2 - FB^2 = BN^2 = (FN + FB) \times (FN - FB)$.

Hence $(FN - FB) : BN :: BN : (FN + FB)$. But (Prop. II.) $FN = VB + VF$; and $FB = VF \sim VB$; therefore $FN - FB = 2\ VB$, or $2\ VF$; and also, $FN + FB = 2\ VF$, or $2\ VB$. Hence $2\ VB : BN :: BN : 2\ VF$; and then, $BN^2 = 4\ VF \times VB$.

Cor. 1. The squares of the ordinates are proportional to their abscissas; since $4\ VF$ is a constant quantity.

Cor. 2. Hence the equation of the curve is $y^2 = p\,x$, p, x and y, being the parameter, abscissa, and ordinate.

PROPOSITION IV.

The parameter [P] *is to the sum of any two ordinates, as their difference is to the difference of the abscissas.*

For $P = 4\ FV$ [Prop. I.]; therefore $P \times VB = BN^2$, and $P \times VH = LH^2$ [Prop. III.]; hence, $P \times BH$, or $P \times MN = LH^2 - BN^2$; therefore, $P : [LH + BN] :: [LH - BN] : MN$; that is, $P : LM :: MO : MN$.

PROPOSITION V.

Any abscissa is to the square of its ordinate, as any right line drawn within the curve parallel to the axis, is to the rectangle of the parts of the double ordinate, which it divides.

For by Proposition IV. $P \times MN = LM \times MO$, and $P \times VB = BN^2$; hence $VB : BN^2 :: MN : LM \times MO$.

Cor. The difference between any two abscissas is directly proportional to the rectangle under the sum and difference of their corresponding ordinates.

PROPOSITION VI.

If from the vertex a right line be drawn through the extremity of an ordinate, so as to meet another ordinate produced, that ordinate will be a mean proportional between the other ordinate and the ordinate produced; that is $HO^2 = HC \times BN$.

For by similar triangles $VB : VH :: BN : HC :: BN^2 : HO^2$ (Prop. III. Cor. 1.); therefore $HO^2 = HC \times BN$.

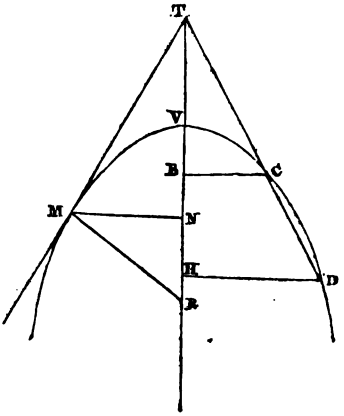
Cor. The abscissa of the produced ordinate, is to the produced ordinate, as the other ordinate to the parameter. Because $HO^2 = P \times VH = HC \times BN$; therefore $VH : HC :: BN : P$.

PROPOSITION VII.

If through the extremities of any two ordinates, a right line be drawn so as to cut the axis; the external part of the axis will be a mean proportional between the abscissas; that is, $TV^2 = VH \times VB$.

By similar triangles, $TB^2 : TH^2 :: BC^2 : HD^2$; but $BC^2 : HD^2 :: VB : VH$, (Prop. III. Cor.); therefore

$TV^2 + 2TV \times VB + VB^2 : TV^2 + 2TV \times VH + VH^2 :: VB : VH$, (4.II.); but by division $TV^2 + 2TV \times$



$VB + VB^2 : 2TV \times VH + VH^2 - 2TV \times VB - VB^2 :: VB : VH - VB$, and by dividing the second and fourth terms by $VH - VB$, we get

$TV^2 + 2TV \times VB + VB^2 : 2TV + VB + VH :: VB : 1$; therefore $TV^2 + 2TV \times VB + VB^2 = 2TV \times VB + VB^2 + VB \times VH$; hence $TV^2 = VB \times VH$.

Cor. 1. Since $VB = \frac{BC^2}{P}$, and $VH = \frac{BC^2}{P}$, (Prop. III.); therefore $VB \times VH = \frac{BC^2}{P} \times \frac{HD^2}{P}$; hence $TV^2 = \frac{BC^2 \times HD^2}{P^2}$; then $TV = \frac{BC \times HD}{P}$; that is, as the parameter is to one of the ordinates, so is the other ordinate to the external part of the axis.

Cor. 2. When D and C coincide, then T D will be a tangent, as T M, and the abscissas will be equal to each other and to V N; therefore as $TV^2 = VB \times VH = VN^2$ (for H and B will coincide at N); hence, $TV =$

V N; that is, the abscissa will be equal to the external part of the axis.

Cor. 3. If from the point of contact M, M R be drawn at right angles to T M, N R is called the sub-normal, which is equal to half the parameter. Because (4. IV.) $NR = \frac{NM^2}{2\,VN}$ $= \frac{1}{2}$ P, (Prop. III.)

Cor. 4. The focus is equi-distant from the point of contact and the intersection of the tangent with the axis.

Because $FM = VF + VN = TF$.

Cor. 5. The focus is equi-distant from the extremity of the sub-normal and the point of contact.

Because $NR = 2\,FV$; therefore $FR = 2\,FV + FN$ $= FV + VN = TF = FM$.

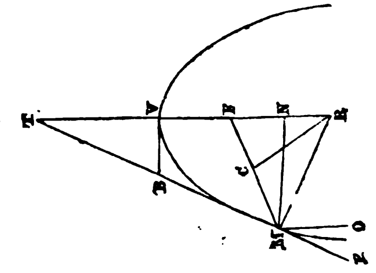
Cor. 6. And hence the focus is the centre of a circle which will pass through T M R.

Cor. 7. If from the point of contact two lines be drawn, the one to the focus and the other parallel to the axis, these lines will make equal angles with the tangent.

For $FM = FT$, (Cor. 4.), the angles F M T and F T M are equal; but the angles O M P and F T M are equal; therefore $OMP = FMT$.

Cor. 8. A perpendicular from the end of the sub-normal to a line drawn from the point of contact to the focus, will cut off from that line a part equal to half the parameter.

Because the triangles R C F and N M F are similar, and have the equal sides F R and F M; therefore $FC = NF$, and consequently $MC = RN = \frac{1}{2}$ P, (Prop. VII. Cor. 3.)



Cor. 9. A tangent at the vertex produced to meet any other tangent, is a mean proportional between half the parameter and half the abscissa.

For, by similar triangles, V B is half of M N, V N being equal T V. But $\frac{1}{4}$ M N^2 = $\frac{1}{2}$ V N $\times$ $\frac{1}{2}p$; therefore V B^2 = $\frac{1}{2}$ V N $\times$ $\frac{1}{2}p$, (Prop. III.)

Cor. 10. Hence the perpendicular V B is a mean proportional between the abscissa and $\frac{1}{4}$ the parameter.

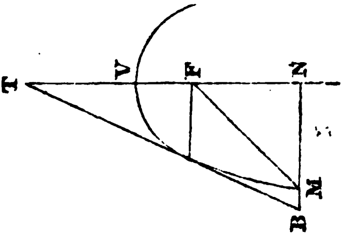
V B^2 = $\frac{1}{2}$ V N $\times$ $\frac{1}{2}$ P = $\frac{1}{4}$ P $\times$ V N.

Cor. 11. Consequently if the points F and B be joined, F B will be at right angles to the tangent, on account of the similarity of the triangles T V B and T F B. Hence it appears that, a perpendicular from the focus to the tangent, and the tangent at the vertex will meet at the same point of the tangent, B.

Cor. 12. A perpendicular from the focus to the tangent, is a mean proportional between the distance of the focus from the point of contact, and the focal distance.

Because B F^2 = F T $\times$ T V, (similar triangles); but F T = F M, (Cor. 4.); therefore F B^2 = F M $\times$ F V.

Cor. 13. If any ordinate be produced to meet the focal tangent, the ordinate so produced will be equal to the distance of the focus from the point of contract.



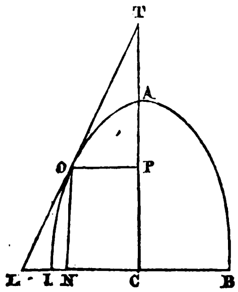
Because M F = V N + V F = V N + V T = N T = (Cor. 4.) N B.

PROPOSITION VIII.

If an ordinate be produced to meet the tangent, then as the double ordinate passing through the point of contact, is to the sum of the two ordinates, so is their difference, to the difference added to the external part of the ordinate produced; that is, 2 P O : B N :: I N : N L.

Because by similar triangles T P : P O :: O N : N L.

but $TP : PO :: 2PO : P$; and $BN : ON :: P : IN$, (Prop. IV.); therefore Ex. Equo. $2PO : P :: ON : NL$; and then Ex. Equo. perturbate, $2PO : BN :: IN : NL$.



Cor. 1. The difference between the ordinates is a mean proportional between double the less and the external part of the lower ordinate. For dividing the terms of the proportion, we get $2PO : BN - 2PO :: IN : NL - IN$; that is, $2PO : IN :: IN : LI$.

Cor. 2. By this means a tangent may be drawn from any given point L, in the ordinate produced. Because $2OP = BI - 2NI$; therefore $BI - 2NI : NI :: NI : LI$; hence $NI^2 + 2NI \times IL = BI \times IL$; therefore $NI^2 + 2NI \times IL + IL^2$, or $NL^2 = BL \times LI$; hence $NL = \sqrt{(BL \times LI)}$ from which it appears that if the point L be given, the points N and O may be found.

Cor. 3. By compounding the terms of the Proposition, $2PO + IN : IN :: BN + NL : NL$; that is, $BN : IN :: BL : NL$.

From the generation of the curve and Prop. II., it appears, that two right lines drawn from any point in the curve, the one to the focus, and the other perpendicular to the directrix, or parallel to the axis, will be always equal; hence the following construction.

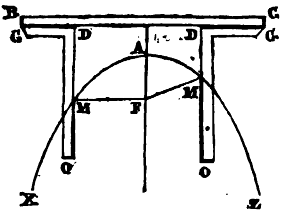
PROPOSITION IX.

PROBLEM.

To construct a parabola by motion.

Provide a ruler, such as B C, and fix it on the plane on which the parabola is to be described; to the directrix B C apply a square O D G, similar to what is commonly called

a carpenter's square, in such a manner that one of its sides D G may lie close to B C; attach one end of a thread, F M O, equal in length to O D, to the end of the ruler at O, the other end of the thread O M F being fixed at F; then



slide the side of the square D G along the ruler B C, keeping the thread stretched by means of a pin M, with its part M O close to the side of the square D O; and the curve A M X described by the motion of the pin, will be one part of a parabola.

If the square be turned over, as represented in the figure, and moved on the other side of the fixed point F, the other part A M Z, of the parabola, will be described by the pin M.

Here O M + M F = O D, and taking away the common part O M, the remainders M D and M F, will always remain equal, which is the property of the curve.

ARITHMETIC OF INFINITIES.

To find the superficial or solid contents of any figure, the pupil is requested to attend to the following preparatory Propositions, showing how to find the sum of certain progressional series.

PROPOSITION I.

In any series of equal numbers, as 1, 1, 1, *&c.*, 2, 2, 2, *&c.*, 3, 3, 3, *&c., the sum will be equal to one of the terms multiplied by the number of terms; that is,* $S = n\,a$, *a, being one of the terms, and n, the number of terms.*

PROPOSITION II.

In a series of numbers in arithmetical progression, beginning with a cypher, and the common difference being 1; *the sum will be equal to half the product of the greater, and number of terms; that is, putting* $g =$ *the greatest term,* $n =$ *the number of terms, and* $S =$ *sum of the series:* $S = \frac{1}{2}\,n\,g$.

$0+1+2+3+4$; then $S = 4 \times 5 \div 2 = 10$. For the reason of this, see *Arithmetic*.

PROPOSITION III.

In a series of squares, whose sides or roots form an arithmetical progression, differing by 1, *and commencing with a cypher; the sum of such a series is equal* $\frac{1}{3}$ *of the greatest term multiplied by the number of terms; when the series is infinitely continued, that is,* $S = \frac{1}{3}\,g^2\,n$.

1. Thus, $0+1+4=5$. But $4\times3=12$; then $\frac{5}{12}=\frac{1}{3}+\frac{1}{12}$.
2. $0+1+4+9=14$. But $9\times4=36$; then $\frac{14}{36}=\frac{1}{3}+\frac{1}{18}$.
3. $0+1+4+9+16=30$. But $16\times5=80$; then $\frac{30}{80}=\frac{1}{3}+\frac{1}{24}$.

In the first series, where the number of terms is 3, the sum exceeds $\frac{1}{3}$ of the greatest term multiplied by the number of terms, by $\frac{1}{12}$; in the second series, where the number of terms is four, the sum exceeds $\frac{1}{3}$ of the greatest term multiplied by the number of terms by $\frac{1}{18}$; in the third series the excess of the sum above $\frac{1}{3}$ of the greatest term multiplied by the number of terms is $\frac{1}{24}$; from which it appears *that, the* excess of the sum of the series above $\frac{1}{3}$ of the product *of the* greatest and number of terms, is continually *diminishing,* according as the number of terms increases;

therefore when the number of terms is infinite, the excess of the sum of the series above $\frac{1}{3}$ of the product of the greatest and number of terms, must necessarily be infinitely small, and consequently less than any assignable quantity; which excess may then be considered as nothing; hence the sum of the series, when the number of terms is infinite, is equal to $\frac{1}{3}$ of the product of the greatest term and number of terms.

PROPOSITION IV.

In a series of cubes, whose roots form an arithmetical progression beginning with a cypher, the common difference being 1, *and the number of terms infinite, the sum will be equal to* $\frac{1}{4}$ *of the product of the greatest term multiplied by the number of terms; that is,* $S = \frac{1}{4} g^3 n$.

1. Thus, $0 + 1 + 8 + 27 = 36$. But $27 \times 4 = 108$; then $\frac{36}{108} = \frac{1}{4} + \frac{1}{12}$.

2. $0 + 1 + 8 + 27 + 64 = 100$. But $64 \times 5 = 320$; then $\frac{100}{320} = \frac{1}{4} + \frac{1}{16}$.

3. $0 + 1 + 8 + 27 + 64 + 125 = 225$. But $125 \times 6 = 750$; then $\frac{225}{750} = \frac{1}{4} + \frac{1}{20}$.

In the first series, the excess of the sum above $\frac{1}{4}$ of the greatest term and number of terms multiplied together, is $\frac{1}{12}$; in the second series, the excess is only $\frac{1}{16}$; and in the third, only $\frac{1}{20}$; therefore, it is obvious that when the number of terms is infinitely great, the excess must necessarily be infinitely small, and therefore, less than any assignable quantity, which excess therefore may be considered as nothing. Hence the truth of the proposition.

PROPOSITION V.

In a series of biquadrates, whose roots form an arithmetical progression, beginning with a cypher, the common difference being 1, *and the number of terms infinite, the sum of such a series will be equal to* $\frac{1}{5}$ *of the product of the greatest term and number of terms multiplied together; that is,* $S = \frac{1}{5} g^4 n$.

The truth of this Proposition may be proved, as in the

foregoing Propositions, by showing that the excess of the sum of the series above the result of the greatest term and number of terms multiplied together, vanishes, when the number of terms becomes infinite.

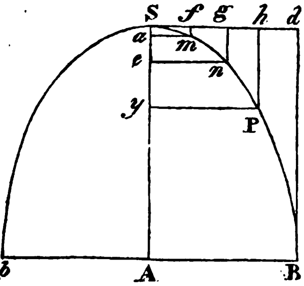
PROPOSITION VI.

In any two ranks of proportionals, having the same number of terms, whether finite or infinite; the first term of one series is to the first term of the other, as the sum of all the terms of the first series to the sum of all the terms of the other. For the truth of this, see Arithmetic.

To apply the preceding Proposition to geometrical quantities, it will be necessary to suppose a line to consist, or to be composed, of an infinite number of points; a surface of an infinite series of lines, either curved or straight; a solid of an infinite series of planes, or superficies.

PROPOSITION VII.

The area of a parabola is equal to $\frac{2}{3}$ of its circumscribing parallelogram.



Draw B d parallel to A S, and S d parallel to A B, conceive S d to be divided into an infinite number of equal parts in the points f, g, h, &c., through which conceive a series of parallels to be drawn, such as $f\,m$, $g\,n$, h P, &c., meeting the semi-ordinates $a\,m$, $e\,n$, y P, &c., in the curve, at the points m, n, P, &c.

Then from the property of the curve, (Prop. III. Cor. 1.) we have the following analogies, viz.

$$S A : A B^2 :: S a : a m^2$$
$$S A : A B^2 :: S e : e n^2$$
$$S A : A B^2 :: S y : y p^2, \text{ \&c.}$$

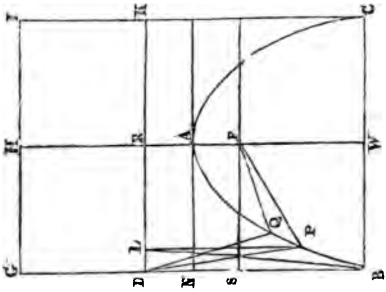
But $S a = f m$, $S e = g n$, $S y = h P$, $S A = d B$; therefore by alternation, we have

$$A B^2 : d B :: y P^2 : h P$$
$$A B^2 : d B :: e n^2 : g n$$
$$A B^2 : d B :: a m^2 : f m, \text{ \&c.}$$

In these proportions, $a m^2$, $e n^2$, $y P^2$, &c., are a series of squares, whose roots $S f$, $S g$, $S h$, &c., are in arithmetical progression, beginning with 0 at S, the common difference being 1, and number of terms infinite; and as the lines $f m$, $g n$, $h P$, &c., are as these squares having $B d$ the greatest term, $S d$ the number of terms; then the sum of all the lines, by Prop. III. *Arithmetic of Infinities*, will be $S = \frac{B d \times S d}{3}$; but $S A \times A B = B d \times S d$; therefore $\frac{S A \times A B}{3} = (S)$ the sum of all the lines $f m$, $g n$, $h P$, &c., which constitute the space $S d B S$, outside the semi-parabola. But the area of the parallelogram $A S d B$ is $S A \times A B$; therefore $S A \times A B - \frac{1}{3} S A \times A B = \frac{2}{3} S A \times A B$ is the area of the semi-parabola $A S P B$; therefore the area of the whole parabola will be equal to $\frac{2}{3} S A \times d B$; but $S A \times d B$ is the area of the circumscribing parallelogram, hence the area of the parabola is $\frac{2}{3}$ of its circumscribing parallelogram.

To prove that the parabola is $\frac{2}{3}$ of the circumscribing parallelogram, take any point P in the curve infinitely near B, (conceive B F to be drawn.) Now, from the nature of the parabola, the angle $P B D$ = angle $P B F$, also $B D = B F$, and $P B$ common; therefore (IV. 1.) the triangles $P B F$ and $P B D$ are equal. Again, the angles $L P Q$ and $F P Q$ are equal; and consequently their supplements are eqnal, viz. $L P B = F P B$, and $L P = P F$, by the property of the parabola, and $P B$ common; therefore the triangles $L P B$ and $F P B$ are equal; hence the triangles $D P B$ and $L P B$ are equal; but the triangle $L P B$ = triangle $D L P$: therefore the triangle $D L P = D P B$.

hence the space D L P B is twice the triangle B L P, and therefore the space D L P B = twice the triangle B P F. In like manner it may be shown that the space D R A Q P B is twice the space B P Q A F B.



Make R H = F W, then the parallelogram G R = twice the triangle B F W; therefore the space B A H G is double of A P B W A, and consequently the parallelogram W G is three times the space W A B; that is, the space A B W is $\frac{1}{3}$ of the parallelogram W G. But G R = W S and S A = A D ∵ G W = 2 W R. Now as the space A B W is $\frac{1}{3}$ G W, it is $\frac{2}{3}$ W R. Hence the parabola B A C is $\frac{2}{3}$ of the parallelogram B K.

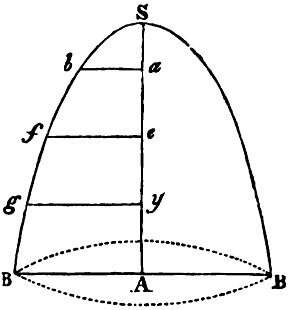
To prove what was quoted in the foregoing demonstration, viz. that the angle P B D is = the angle P B F. From the nature of the parabola P T = P L and the angle D P L being infinitely small causes no sensible difference between P D and P L ∴ P D = P F and B F is common in the two triangles; also B D = B F ∴ (8.I.) the angle P B F = P B D.

PROPOSITION VIII.

Every parabolic conoid is equal to half its circumscribing cylinder.

If the semi-parabola B S A be made to revolve about its *axis S A, the solid* thus formed is called a parabolic conoid, *and may be* conceived to be constituted of an infinite series

of circles parallel to its base B B.



From the property of the curve, Prop. III. S A : A B :: A B : P $\left(= \frac{A B^2}{S A}\right)$ the parameter.

Then $S a \times P = b a^2$
$S e \times P = f e^2$
$S y \times P = g y^2$

Here $S a$, $S e$, $S y$, &c., are a series in arithmetical progression, then $S a \times P$, $S e \times P$, $S y \times P$, &c. are in arithmetical progression, [GREGORY's *Philosophy of Arithmetic.*] Therefore $b a^2, f e^2, g y^2$; &c., are a series in arithmetical progression, beginning at S, the first term being 0, the common difference 1, $A B^2$, the greatest term, and S A the number of terms. Therefore $A B^2 \times \frac{1}{2} S A = S$, the sum of the series, (Prop. II. *Arithmetic of Infinities.*) But the areas of all the circles, which constitute the solid, and whose radii are $b a$, $f e$, $g y$, &c., are, by Prob. XVIII. Sec. II., equal to $2 b a^2 \times n$, $2 f e^2 \times n$, $2 g y^2 \times n$, n being equal to ·7854; therefore, putting $d = 2$ A B, and $h =$ S A, the sum of all the circular areas constituting the parabolic conoid will be $\frac{1}{2} n d^2 h$. But $n d^2 h$ is the content of the cylinder, the diameter of whose base is d, and height h, (Prob. IV. Sec. 4); therefore the parabolic conoid is half of its circumscribing cylinder.

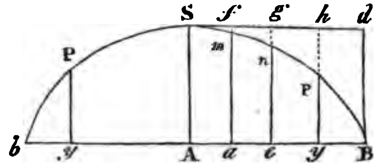
Cor. The solidity of the lower frustrum of a conoid cut off by a plane parallel to the base, is equal to half the sum of the areas of both bases multiplied by the height of the frustrum.

It has been shown in the Proposition, that the areas of the circles which constitute the frustrum are a series in arithmetical progression, the sum of which is equal to $\frac{1}{2}$ the sum of the extremes multiplied by the number of terms; but the extremes are the areas of the two bases of the frustrum, and the height the number of terms; therefore $S = \frac{1}{2} (A + a) \times h$, S being the solidity, A the area of the greater base, a the area of the less base, and h the height of the frustrum.

PROPOSITION IX.

Every parabolic spindle is equal to $\frac{8}{15}$ of its circumscribing cylinder.

Draw Sd parallel to A B, and also fa, ge, hy, &c., parallel to A S. It has been proved in Proposition X., that the lines fm, gn, hP, &c., are a series of squares, whose roots



form an arithmetical progression; therefore their squares, viz. fm^2, gn^2, $h\mathrm{P}^2$, will be a series of biquadrates, whose roots will form an arithmetical progression.

Now, the spindle is generated by conceiving the segment C S B to revolve about b B; and the solid itself is composed of the series of circles whose radii are ma, ne, Py, &c.

Again, $\mathrm{SA} - fm = ma$
$\mathrm{SA} - gn = ne$
$\mathrm{SA} - h\mathrm{P} = \mathrm{P}y$, &c.; therefore by squaring,

1. $\mathrm{SA}^2 - 2\,\mathrm{SA} \times fm + fm^2 = ma^2$
2. $\mathrm{SA}^2 - 2\,\mathrm{SA} \times gn + gn^2 = ne^2$
3. $\mathrm{SA}^2 - 2\,\mathrm{SA} \times h\mathrm{P} + h\mathrm{P}^2 = \mathrm{P}y^2$, &c.

In these equations SA^2, SA^2, SA^2, &c., form a series of equal squares, of which A B is the number of terms, therefore their sum will be $\mathrm{SA}^2 \times \mathrm{AB}$.

And because fm, gn, hP, &c., are a series of squares, of which S A is the greatest term, and A B the number of terms; their sum will be $\frac{\mathrm{SA} \times \mathrm{AB}}{3}$, (Prop. III. *Arithmetic of Infinities*,) which being multiplied by 2 S A, will give the sum of that part of the equation, $2\,\mathrm{SA} \times fm$, $2\,\mathrm{SA} \times gn$, $2\,\mathrm{SA} \times h\mathrm{P}$, &c., viz. $\frac{2\,\mathrm{SA}^2 \times \mathrm{AB}}{3}$.

Again, fm^2, gn^2, hP^2, &c., are a series of terms of biquadrates, as has been shown above, whereof dB^2, or SA^2 is the greatest, and AB the number of terms; therefore their sum (by Prop. V. *Arithmetic of Infinities*) will be $\frac{SA^2 \times AB}{5}$. Hence it is obvious, that the sum of ma^2. ne^2, Py^2, &c., will be

$$SA^2 \times AB - \frac{2SA^2 \times AB}{3} + \frac{SA^2 \times AB}{5} =$$

$$\frac{6SA^2 \times AB}{5} - \frac{2SA^2 \times AB}{3} = \frac{8SA^2 \times AB}{15}.$$

But the areas of the circles, whose radii are SA, ma, ne, Py, &c., are found by multiplying the squares of their diameters by ·7854 ($= n$); therefore, the sum of double such a series of circles is $\frac{8nD^2H}{15}$, putting $D = 2SA$, $H = 2AB =$ the solidity of the whole spindle.

But the solidity of the circumscribing cylinder is nD^2H; therefore the solidity of the spindle, viz. $\frac{8nD^2H}{15}$ is the eight-fifteenth of its circumscribing cylinder.

Cor. From this may be derived a method of finding the the solidity of the frustrum, $SAyp$, of a spindle.

The area of a circle whose radius is SA being the greatest term, and the area of the circle whose radius is Py, the least term, and Ay the number of terms; then the sum of such a series, that is, the sum of all the circles included between A and y will be the solidity of the required frustrum.

From what has been shown in the Proposition, the sum of all the series SA^2, ma^2, gn^2, Py^2, is $\left(SA^2 - \frac{2SA \times hp}{3} + \frac{hp^2}{5}\right) \times Ay = Z$. By multiplying the equa-

tion by 3, we get $\left(3\,SA^2 - 2\,SA \times hp + \frac{3\,hp^2}{5}\right) \times Ay = 3\,Z$.

Divide both sides of the equation by Ay, and $3\,SA^2 - 2\,SA \times hp + \frac{3\,hp^2}{5} = \frac{3\,Z}{Ay}$; but $SA^2 - 2\,SA \times hp = py^2 - hp^2$, (by the Proposition); then the difference of these will be equal, that is, $2\,SA^2 + \frac{3\,hp^2}{5} = \frac{3\,z}{Ay} - py^2 + hp^2$, and by transposition, $2\,SA^2 + Py^2 - \frac{2}{5}hp^2 = \frac{3\,z}{Ay}$; divide by $\frac{3}{Ay}$, and we get $(2\,SA^2 + py^2 - \frac{2}{5}\,hp^2)\ \frac{1}{3}\,Ay = z$. Then the areas of all the circles included between A and y will be $(1{\cdot}5708 \times 2\,SA^2 + 3{\cdot}1416 \times py^2 - 3{\cdot}1416 \times \frac{2}{5}hp^2) \times \frac{1}{3}\,Ay = ({\cdot}5236 \times 2\,SA^2 + 1{\cdot}0472 \times py^2 - 1{\cdot}0472 \times \frac{2}{5}hp^2) \times Ay =$ the solidity; put $D = 2\,SA$, $C = 2\,py$, $d = 2\,hp$, and $L = Ay$; then $(2\,D^2 + C^2 - \frac{4}{10}\,d^2) \times L \times {\cdot}2618$ is the solidity of the frustrum.

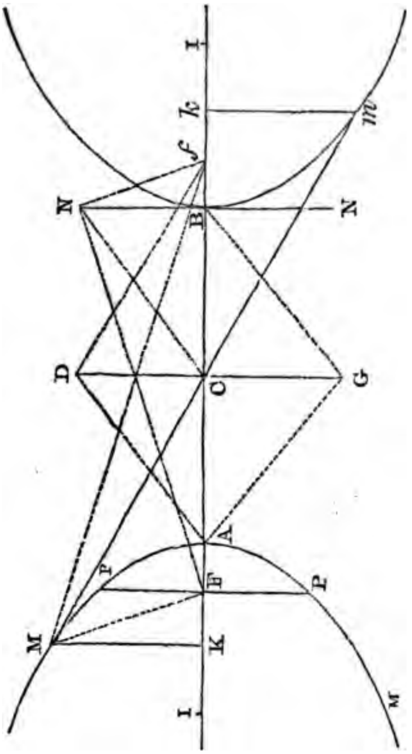
OF THE HYPERBOLA.

DEFINITIONS.

1. If in the line B C produced both ways, there be assumed two points F and f, equi-distant from B C; and if from the centres F and f, with the radii B I and C I (I being assumed beyond B), arcs be described so as to intersect each other in M, and an infinite number of such points be found; the curve passing through these points is called a hyperbola.
2. The points F and f are called the foci.
3. The line B C is called the transverse or greater axis.
4. *The point* O *in* the middle of B C is the centre.

5. The line D G passing through the centre perpendicular to the axis, and of such a length that B D may be equal to O F, is called the conjugate or less axis.

6. Any line passing through the centre and terminated both ways by the curve, is called a diameter.



7. Diameters are called conjugates, when one of them is parallel to a tangent passing through the vertex of the other.

8. A right line drawn from the centre to a diameter produced, and parallel to the tangent at its vertex, is called an ordinate; and if the ordinate be perpendicular to the diameter, it is said to be rightly applied.

9. The part of the axis, or of any diameter produced, which is intercepted between the vertex of that diameter and the ordinate, is called an abscissa.

10. A third proportional to any diameter and its conjugate is called the parameter of that diameter.

PROPOSITION I.

The difference of right lines drawn from any point in the curve to the foci, is equal to the transverse axis, and therefore always equal.

For $f\mathrm{M} = \mathrm{CI}$, and
$\mathrm{FM} = \mathrm{BI}$; therefore $f\mathrm{M} - \mathrm{FM} = \mathrm{CI} - \mathrm{BI} = \mathrm{BC}$.

Cor. 1. Hence it appears that the curve must pass through B and C.

$$\text{For } f\mathrm{B} - \mathrm{FB} = \mathrm{FC} - f\mathrm{C} = \mathrm{BC}.$$

Cor. 2. By the definition B C is bisected in O, and $\mathrm{BF} = \mathrm{C}f$; therefore the focal distance is bisected in O.

PROPOSITION II.

Half the conjugate axis is a mean proportional between the distance of the focus from the extremities of the transverse axis.

For $\mathrm{BD}^2 - \mathrm{BO}^2 (= \mathrm{OD}^2) = \mathrm{FO}^2 - \mathrm{BO}^2 = \mathrm{FB}^2 + \mathrm{BO}^2 + 2\,\mathrm{FB} \times \mathrm{BO} - \mathrm{BO}^2 = \mathrm{FB}^2 + 2\,\mathrm{FB} \times \mathrm{BO} = (\mathrm{FB} + 2\,\mathrm{BO}) \times \mathrm{FB} = \mathrm{FC} \times \mathrm{FB}$; that is, $\mathrm{OD}^2 = \mathrm{FC} \times \mathrm{FB}$. See the last figure.

PROPOSITION III.

If the conjugate axis be applied to the vertex of the transverse axis, the centre of the hyperbola will be the centre of a circle which will pass through the foci and the extremity of the conjugate.

For by the construction $\mathrm{ON} = (\mathrm{BD})\ \mathrm{OF} = \mathrm{O}f$; therefore if from the centre O, with the radius O F, a circle be

described, it will pass through N and f.—See the last figure.

Cor. Therefore the distance of either focus from the extremity of the conjugate, is a mean proportional between the distance of the foci, and the distance of that focus from the vertex to which the conjugate is applied. For, as a circle, with the centre O, passes through $F N f$, the angle $F N f$ is right; therefore $f N^2 = F f \times f C$, and $F N^2 = F f \times F C$, (8. IV.)

PROPOSITION IV.

The parameter is double the ordinate applied to the focus.—See last figure.

For $P F^2 + F f^2 = P f^2$; but $F f = C B + 2 B F$, and $P F = C B + P F$. Then $P f^2 = C B^2 + C B \times 2 P F + P T^2$, and $P f^2 = P T^2 + f F^2$; therefore $C B \times 2 P F = f F^2 - C B^2 = 4 F O^2 - 4 B O^2 = 4 B D^2 - 4 B O^2 = 4 O D^2 = D G^2$; that is, $C B \times 2 P F = D G^2$, $C B : D G :: D G : 2 P F$; then by definition 10, $2 P F$ ($= P P$) is the parameter.

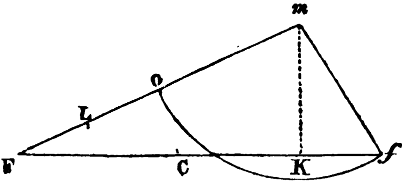
Cor. The distance between the foci is a mean proportional between the transverse, and the sum of the transverse and parameter.

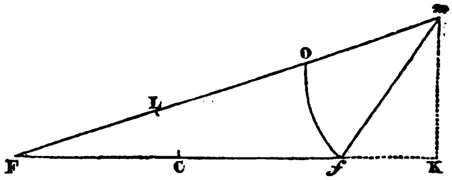
For $F f^2 = P f^2 - P F^2 = (C B + 2 P F) \times C B$; therefore $C B : F f :: F f : (C B + 2 P F)$.

LEMMA.

In any plane triangle, if the base and difference of the sides be bisected, and a perpendicular be let fall from the vertical angle to the base; then half the base is to the sum or difference of half the difference of the sides and one of them; as half the difference of the sides to the distance of the middle of the base from the perpendicular.—(See *Trigonometry.*)

It is easily proved that $F f : F m + f m :: F O : F f + 2 f K$; hence $F C : f m + F L$, or $F m - F L :: F L : F C + f K$, or $C K$.





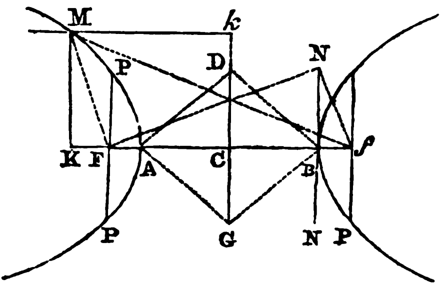
Cor. Hence, $fm = \frac{FC \times CK}{FL} - FL$ and

$Fm = \frac{FC \times CK}{FL} + FL.$

PROPOSITION V.

As the square of half the transverse to the rectangle of the focal distances from the vertices; so is the rectangle of the abscissa and sum of the transverse and abscissa, to the square of the ordinate; that is, $AC^2 : AF \times FB :: AK \times KB : mK^2$.

By the lemma $AC : FC :: CK : fm - AC$; which being squared and divided will be, $AC^2 : FC^2 - AC^2 :: CK^2 : fm - 2fm \times AC + AC^2 - CK^2$; which being inverted and divided, will be $AC^2 : FC^2 - AC^2 :: CK^2 - AC^2 : fm^2 - 2fm \times AC + 2AC^2 - CK^2 - FC^2$. But $FC^2 - AC^2 = AF \times FB$; $CK^2 - AC^2 = AK \times KB$; and $2fm \times AC - 2AC^2 = 2FC \times CK$, (by the last Cor.), $fm^2 - 2fm \times AC - 2AC^2 - CK^2 - FC^2 = fm^2 - fK^2 = mK^2$; hence $AC^2 : AF \times FB :: AK \times KB : mK^2$.



Cor. The square of half the transverse is to the square of half the conjugate, as the rectangle of the abscissa, and the sum of the abscissa and transverse, to the square of the ordidate.

For $CD^2 = AF \times FD$.

Cor. 2. $DG^2 = AB \times P$.

Cor. 3. Hence the squares of the ordinates are proportional to the rectangles of the abscissa by the sum of the abscissa and transverse.

Cor. 4. The square of half the conjugate is to the square of half the transverse, as the sum of the squares of half the conjugate and the distance of the ordinate from the centers to the square of the ordinate of the conjugate.

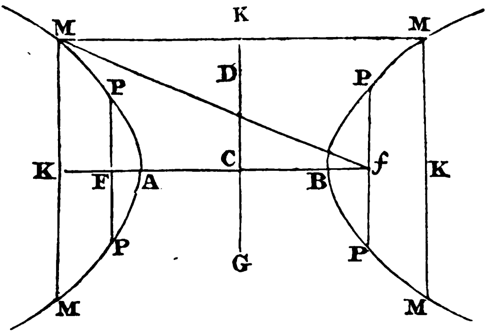
By Cor. 1, Proposition V., $AC^2 : CD^2 :: CK^2 - AC^2 : MK^2$; hence by inversion and division $CD^2 : AC^2 :: MK^2 + CD^2 : CK^2$, or MK^2.

Cor. 5. As the conjugate is to the parameter, so is the sum of the squares of half the conjugate and the distance of the ordinate from the centre to the square of the ordinate of the conjugate.

Cor. 6. Therefore these squares are as the squares of the ordinates of the conjugate.

Cor. 7. As $CD^2 = AF \times FB$; then $AF \times FB : AC^2 :: CD^2 + MK^2 : MK^2$.

Cor. 8. And as $AC^2 = FC^2 - CD^2$; then $AK \times KB : CF^2 - CD^2 :: MK^2 : CD^2$.



Cor. 9. Also, as $AB^2 = \frac{GD^4}{P^2}$; then $GD^2 : P^2 :: AK \times KB : MK^2 :: MK^2 : MK^2 + CD^2$.

PROPOSITION VI.

If through the extremities of any two ordinates, a right line be drawn so as to meet the axis; then as the square of the distance of one of the ordinates from the point of intersection, is to the rectangle of the abscissa by the abscissa and transverse; so is the sum of the distances of both ordinates from the point of intersection, to the sum of their distances from the centre; that is, $TQ^2 : AQ \times QB :: TQ + TL : CQ + CL$.

For, by similar triangles, and Proposition V., $TQ^2 : AQ \times QB :: TL^2 : AL \times LB$; therefore by division $TQ^2 : AQ \times QB :: (TL^2 - TQ^2 : AL \times LB - AQ \times QB ::) CL^2 - CQ^2 - 2TC \times CL + 2TC \times CQ : CL^2 - CQ^2$, and by dividing the two last terms by $CL - CQ$, we shall have $TQ^2 : AQ \times QB :: (CL + CQ - 2CT : CL + CQ ::) TL + TQ : CL + CQ$.

Cor. If P and M coincide; then A Q and A L will become equal to A N, and the line T M will be a tangent, as *T O, therefore* $TN^2 : AN \times NB :: 2TN : 2CN$; hence

by dividing the two last by 2, and the first and third by TN, we shall get CN : BN :: AN : TN; which is the property of the tangent.

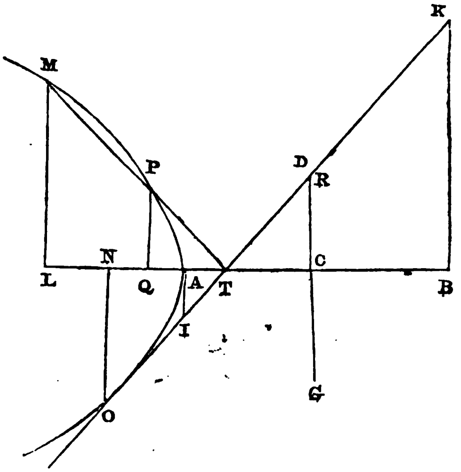
Cor. 2. By dividing the last analogy, we shall get CN : BN − CN :: AN : TN − AN; that is, CN : AC :: AN : AT.

Cor. 3. By inverting and dividing the first, we shall get, CN : CN − AN :: BN : BN − TN; that is, CN : AC :: BN : BT.

Cor. 4. Therefore Ex. Equo. AN : AT :: BN : BT.

Cor. 5. By dividing the third, we shall get CN : AC :: CN − AN : AC − AT; that is, CN : AC :: AC : CT.

Cor. 6. By inverting and compounding the last analogy, we get CT : CA :: CT + CA : CA + CN; that is, CT : CA :: BT : BN.



Cor. 7. By inverting and dividing the 5th Cor., we get CT : CA :: (AC − CT : CN − AC ::) AT : AN.

Cor. 8. By inverting and dividing the 6th Cor., we get CT : BT :: AC − CT : BN − BT; that is, CT : BT :: AT : NT.

Cor. 9. By comparing the 6th and 8th Cors. then by Ex. Equo. A C : B N : : A T : N T.

Cor. 10. By comparing the 7th and 8th Cors. then Ex. Equo. A C : A N : : B T : N T.

Cor. 11. If the point N be at an infinite distance, then A C will be equal to A T, and therefore the tangent passes through the centre.

Cor. 12. If perpendiculars to the extremities of the transverse and semi-conjugate be produced to meet the tangent produced; then A I : O N : : C R : B K, (Cor. 8. and similar triangles.)

Cor. 13. By similar triangles, T B : T N : : B K : N O; therefore by the 10th Cor. A C : A N : : B K : N O.

Cor. 14. By similar triangles A T : T N : : I A : N O; therefore by Cor. 9, A C : B N : : A I : N O.

Cor. 15. By Ex. Equo. perturbate, A N : B N : : A I : B K.

Cor. 16. By compounding the 13th and 14th Cors. AC^2 : $AN \times NB$: : $AI \times BK$: NO^2 : : (Prop. V. Cor. 1.) $CD^2 : NO^2$; therefore $CD^2 = IA \times BK = AF \times FB$.

Cor. 17. When the tangent passes through the centre, then will A I and B K be equal; and therefore C B = A I = B K = B V, (Cor. 16.)

Cor. 18. Therefore if half the conjugate be applied to the vertex of the transverse diameter, a tangent at an infinite distance will pass through the centre and the extremity of the conjugate.

Cor. 19. Therefore if a right line be thus drawn, it will continually approach the curve, but will never meet it; being a tangent at an infinite distance.

PROPOSITION VII.

A solid formed by the rotation of an hyperbola upon its axis, is to the circumscribing cylinder as half the transverse + $\frac{1}{3}$ the abscissa, to the sum of the transverse and abscissa.

Let t and x be the transverse and abscissa, P the para-

meter, n a quantity, by which if the square of any radius be multiplied, the product will be the area of the circle.—See Fig. Prop. V.

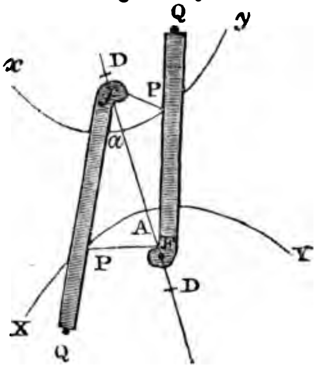
By Prop. V. $MK^2 = \frac{P}{t} \times (tx + x^2)$; therefore the area of a circle whose radius is M K, will be $\frac{nP}{t} \times (tx + x^2)$, and the sum of all the circles between A and P must be $\frac{nPx^2}{t} \times (\frac{1}{2}t + \frac{1}{3}x)$, *Arithmetic of Infinities;* but the solidity of the circumscribing cylinder $\frac{nPx^2}{t} \times (t + x)$ therefore the hyperbolic conoid is to the cylinder as $\frac{1}{2}t + \frac{1}{3}x$ to $t + x$.

Cor. 1. $3t + 2x$ will be to $6t + 6x$ in the same ratio.

Cor. 2. When the abscissa becomes equal to the transverse, the conoid will be to the cylinder as 5 to 12.

Besides the method given at the beginning of this Chapter to construct a hyperbola by means of points, the following may be found useful in practice;—

Fasten one end of a long ruler fM Q at the point f by



means of a pin on a plane, so as to turn freely about f as a centre. Then fasten the end of a thread F M Q, shorter than the ruler, at the point F, and the other end of the thread at the end Q of the ruler.

Then turn the ruler fM Q about the fixed point f, at the same time, keeping the thread tight, and its part M Q close to the side of the ruler, by means of the pin P; the curve line A x described by the pin P is one part of an hyperbola.

If the ruler be turned, and moved on the other side of F, the other part A y of the same hyperbola will be described.

If one end of the thread be fixed to the end Q of the ruler, while the other end of the ruler is fixed at F, the hyperbola $x\,a\,y$ is described.

CONIC SECTIONS.

SECTION IV.

PROBLEM I.

The transverse and conjugate diameters of an ellipsis being given, to find the area.

RULE. Multiply the transverse and conjugate diameters together, and the product arising by ·7854, and the result will be the area.*

1. Let the transverse axis be 35, and the conjugate axis 25; what is the area?

Ans. $35 \times 25 \times \cdot 7854 = 687 \cdot 225$.

2. The longer diameter of an ellipse is 70, and the shorter 50; required the area? *Ans.* 2748·9.

3. What is the area of an ellipse whose longer axis is 80, and shorter axis 60? *Ans.* 3769·92.

4. What is the area of an ellipse, whose diameters are 50 and 45? *Ans.* 1767·15.

PROBLEM II.

To find the area of an elliptical ring.

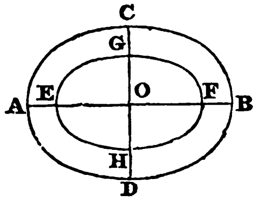
RULE. Find the area of each ellipse separately, and their difference will be the area of the ring.

---

* *Demonstration.* By Proposition XII. Cor. 2, the diameter of a circle equal in area to an ellipse, is a mean proportional between the transverse and conjugate diameters of the ellipse; thus, putting d for the diameter of the circle, t for the transverse, and c for the conjugate diameter of the ellipse; then $t : d :: d : c$, and $d^2 = t \times c$; but $d^2 \times \cdot 7854 =$ area of the circle, Prob. XXVIII. Sec. 2, which is equal to the area of the ellipse; therefore $t \times c \times \cdot 7854 =$ area of the ellipse.

Or, From the product of the two diameters of the greater ellipse deduct the product of the diameters of the less, and multiply the remainder by ·7854 for the area of the ring.*

1. The transverse diameter A B is 70, and the conjugate C D 50; and the transverse diameter E F of another ellipse,



having the same centre O, is 35, and the conjugate G H is 25; required the area of the elliptical space between their circumferences?

$70 \times 50 \times \cdot 7854 = 2748 \cdot 9$; and $35 \times 25 \times \cdot 7854 = 687 \cdot 225$; then $2748 \cdot 9 - 687 \cdot 225 = 2061 \cdot 675 =$ area of the elliptical ring.

$$\begin{array}{r} 70 \times 50 = 3500 \\ 35 \times 25 = 875 \\ \hline 2625 \times \cdot 7854 = 2061 \cdot 675 = \text{area.} \end{array}$$

2. The transverse and conjugate diameters of an ellipse are 60 and 40, and of another 30 and 10; required the area of the space between their circumferences? *Ans.* 1649·34.

3. A gentleman has an elliptical flower garden, whose greater diameter is 30, and less 24 feet; and has ordered a gravel walk to be made round it of 5 feet 6 inches in width, required the area of the walk? *Ans.* 561·561 feet.

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* *Demonstration.* The first Rule is self-evident; for the space E G F H being deducted from A C B D, the remainder will be the space between both circumferences.

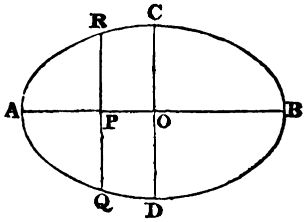
The second Rule is equally evident. For, by putting T and C for the transverse and conjugate diameters of the larger ellipse, its area is $T \times C \times \cdot 7854$, and the area of the smaller ellipse is $t \times c \times \cdot 7854$; then their difference is $T \times C \times \cdot 7854 - t \times c \times \cdot 7854 = (T \times C - t \times c) \times \cdot 7854$ which is the rule.

PROBLEM III.

Given the height of an elliptical segment, whose base is parallel to either of the axes of the ellipse, and the two axes of the ellipsis, to find the area.

RULE. Divide the height of the segment by that diameter of which it is a part, to three places of decimals, find the quotient in the column height of the Table, and take out the corresponding area segment. Multiply the area segment thus found and both the axes of the ellipsis together, and the result will give the area required.*

1. Required the area of an elliptical segment R A Q,



whose height A P is 20; the transverse axis A B being 70, and the conjugate axis C D 50?

$20 \div 70 = \cdot 285\frac{5}{7} =$ the tabular versed sine, the corresponding segment answering to which, is ·185166; then $\cdot 185166 \times 70 \times 50 = 648 \cdot 081$, the area.

2. What is the area of an elliptical segment cut off by a

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* *Demonstration.* By Corollaries to Proposition XII. an ellipse is to the rectangle of its two axes, as any circle is to the square of its diameter. And also any segment of an ellipse is to a like segment of a circle, as the rectangle contained by the two axes of the ellipse is to the square of the diameter of the circle. But by Prob. XXVIII. Rule 3, the segment of the circle is found by multiplying the area segment as found in the Table of circular segments, corresponding to the height divided by the square of the diameter; therefore, the segment of the ellipse is equal to the product of both its axes multiplied by the area segment corresponding to the height of the segment divided by the diameter of which the given height is a part.

chord parallel to the shorter axis, the height of the segment being 10, and the two diameters 35 and 25?

Ans. 162·02025.

3. What is the area of an elliptical segment cut off by a chord parallel to the longer axis, the height of the segment being 10, and the two diameters 40 and 30?

Ans. 275·0064.

4. What is the area of an elliptical segment cut off by a chord parallel to the shorter diameter, the height being 10, and the two diameters 70 and 50? *Ans.* 240·884.

PROBLEM IV.

To find the circumference of an ellipse, by having the two diameters given.

RULE. Multiply the sum of the two diameters by 1·5708, and the product will give the circumference nearly; that is, putting t for the transverse, c for the conjugate, and p for 3·1416; the circumference will be $(t + c) \times \frac{1}{2}p$.*

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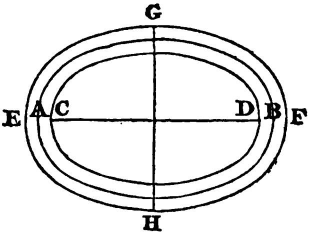
* *Demonstration.* It has been shown that the geometrical mean between the two axes is equal to the diameter of a circle equal in area to the ellipse; but the arithmetical mean exceeds this geometrical mean, while the circumference of the ellipse exceeds that of the circle equal in area to it; therefore $\frac{(t+c)}{2} \times p = (t+c) \times \frac{1}{2}p$ will give the circumference nearly.



$\frac{(t+c)}{2} \times p$ is greater than the circumference of a circle; but

1. Let the transverse axis be 24, and the conjugate 18; required the area?

$(24 + 18) \times 1{\cdot}5708 = 42 \times 1{\cdot}5708 = 65{\cdot}9736$ is the circumference nearly.

2. Required the circumference of an ellipse whose transverse axis is 30, and conjugate 20? *Ans.* 78·54.

3. Required the circumference of an ellipse whose diameters are 60 and 40? *Ans.* 157·08.

4. What is the circumference of an ellipse whose diameters are 6 and 4? *Ans.* 15·708.

5. What is the circumference of an ellipse whose diameters are 3 and 2? *Ans.* 7·854.

PROBLEM V.

To find the length of any arc of an ellipse.

RULE. Find the length of the circular arc xy, intercepted by O C, O B, and whose radius is half the sum of

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$\sqrt{(t \times c)} \times p$ is less than the circumference of the ellipse equal in area to the circle; therefore $(t + c) \times \frac{1}{2} p$ is the circumference of the ellipse nearly.

Otherwise:

Let A B be the curve whose length we require; and let two other curves E F and C D be described at equal but very small distances from it, the one without it, and the other within it; these two curves do not differ much from ellipses, and the difference of their areas, as found by Prob. II. Sec. IV. will give the area of the ring or space between them nearly; but this area is equal to the curve A B multiplied by E C; the distance between the two curved E F and C D; therefore the area of the ring divided by E C will give the length of the curve A B nearly enough for practical purposes.

Therefore, putting $t =$ the transverse axis of the ellipse, whose length is required,

$c =$ its conjugate
$p = 3{\cdot}1416$
$d = \frac{1}{2}$ E C $=$ A C or A E.

Then $(t + 2d) \times (c + 2d) \times \frac{1}{4} p\ (= {\cdot}7854) =$ the area of the curve E F.

And $(t - 2d) \times (c - 2d) \times \frac{1}{4} p =$ the area of the curve C D.

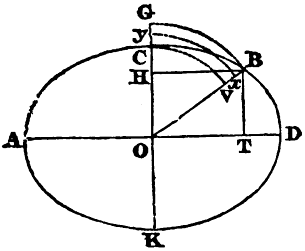
The difference $(t + c) \times p d =$ the area of the ring.

Therefore $(t + c) \times p d \div 2 d =$ the length of the curve A B; that is, $(t + c) \times \frac{1}{2} p =$ the length of the curve.

O C, O B; and it will be equal to the elliptical arc B C nearly.*

1. Let the axis A D be 24, C K 18, and O T 3; required the length of the arc B C?

Here we have $TD = 9$, and $AT = 15$; then from the property of the ellipse, we have $AO^2 : OC^2 :: AT \times$



$$TD : TB^2 = \frac{9^2 \times 9 \times 15}{12 \times 12} = \frac{9 \times 9 \times 15}{16}, \text{ and } OB =$$

$$\sqrt{(OT^2 + TB^2)} = \sqrt{(9 + \frac{9 \times 9 \times 15}{16})} = 9{\cdot}21616, \text{ the}$$

radius of the circle of which G B is an arc; but O C is the radius of the circle of which C V is an arc; therefore the radius of the circle of which xy is an arc, is $\frac{1}{2}OC + \frac{1}{2}OB = 9{\cdot}10808$. But by *Trigonometry* $zB \div OB = 3 \div 9{\cdot}21616 = {\cdot}325515$ which is the sine of the angle C O B, or arc $x\,y$, to the radius 1, answering to $18{\cdot}9968$ degrees. Therefore by Problem XVII., Rule 1, the length of the arc

---

* *Demonstration.* It is manifest that the circular arc xy is an arithmetical mean between the circular arcs C V and G B; but xy is nearly equal to the elliptical arc C B. Hence the rule is evident.

Note. The nearer the axes of the ellipse approach towards equality, the more exact the result of the operation by this rule; and the less the elliptical arc, the nearer its exact length will the arc xy approach.

This rule is the easiest for practice. Other rules might be given which would find more accurate results; but being both tedious and difficult, and the investigation necessarily involving the fluxional or *differential calculus,* they are omitted.

xy is ·01745 × 18·9968 × 9·10808 = 3·0192, which is also equal to the length of the elliptical arc C B nearly.

2. Given A D 30, C K 20, and O T 5; required the length of the arc B C? *Ans.* 5·03917786255.

3. Given A D 40, C D 30, and O T 5; required the length of the arc B C? *Ans.* 5·033880786.

PROBLEM VI.

Given the diameters and abscissas, to find the ordinate.

RULE. Say, as the transverse is to the conjugate, so is the square root of the rectangle of the two abscissas, to the ordinate.*

1. In the ellipse A C D K, the transverse diameter A D is 100, the conjugate diameter C K 80, and the abscissa B T 10; required the length of the ordinate T B?

$100 : 80 :: \sqrt{(90 \times 10)} : TB = 24$. (See the last figure.)

2. Let the transverse axis be 35, the conjugate 25, and the abscissa 7; required the ordinate? *Ans.* 10.

3. Given the two diameters 70 and 60, and the abscissa 10; required the ordinate? *Ans.* 20·999.

PROBLEM VII.

Given the transverse axis, conjugate, and ordinate, to find the abscissas.

RULE. As the conjugate is to the transverse diameter, so is the square root of the difference of the squares of the ordinate and semi-conjugate, to the distance between the ordinate and centre. Then this distance being added to,

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* *Demonstration.* The equation of the curve is $d^2 : c^2 :: x(d-x) : y^2$; putting d for the transverse diameter, c for its conjugate, and x for an abscissa to the ordinate y; then $d : c :: \sqrt{(x(d-x))} : y$, which is the rule.

From this equation expressing the property of the curve, the following rules are derived.

and subtracted from, the semi-diameter, will give the two abscissas.*

1. Let the diameters be 35 and 25, and the ordinate 10, required the abscissas?

By the Rule $\frac{35}{2} \pm \frac{35}{25} \sqrt{\left(\left[\frac{25}{2}\right]^2 - 10^2\right)} = \frac{35 \pm 21}{2} = 28$ and 7, the two abscissas.

2. Let the diameters be 120 and 40, and the ordinate 16, required the abscissas? *Ans.* 96 and 24.

PROBLEM VIII.

Given the conjugate axis, ordinate, and abscissas, to find the transverse axis.

RULE. Find the square root of the difference of the squares of the semi-conjugate axis and the ordinate, which added to, or subtracted from, the semi-conjugate, according as the less abscissa or greater is given.

Then say, as the square of the ordinate is to the rectangle of the conjugate and the abscissa, so is the sum or difference found above to the transverse required.†

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* *Demonstration.* From the property of the curve, we have $d^2 : c^2 :: x(d - x) : y^2$, then $c^2 dx - c^2 x^2 = d^2 y^2$, and dividing c^2, we get $dx - x^2 = \frac{d^2 y^2}{c^2}$, and $x^2 - dx = -\frac{d^2 y^2}{c^2}$, and by completing the square, we get $x^2 - dx + \frac{d^2}{4} = \frac{d^2}{4} - \frac{d^2 y^2}{c^2} = d^2 \left(\frac{1}{4} - \frac{y^2}{c^2}\right) = d^2 \left(\frac{1}{4} \frac{c^2 - y^2}{c^2}\right) = \frac{d^2}{c^2} (\frac{1}{4} c^2 - y^2)$; then $x - \frac{d}{2} = \pm \sqrt{\left(\frac{d^2}{c} (\frac{1}{4} c^2 - y^2)\right)} = \pm \frac{d}{c} \sqrt{(\frac{1}{4} c^2 - y^2)}$, and $x = \frac{d}{2} \pm \frac{d}{c} \sqrt{(\frac{1}{4} c^2 - y^2)}$ which corresponds with the rule.

† *Demonstration.* The same notation being retained, we have, from the property of the curve $d^2 : c^2 : (xd - x^2) :: y^2$, and therefore $y^2 d^2 = c^2 x d - c^2 x^2$; then by ransposition, $y^2 d^2 - c^2 x d = - c^2 x^2$,

1. Let the ordinate be 10, and the less abscissa 7; what is the diameter, allowing the conjugate to be 25?

$$\sqrt{\left(\left[\frac{25}{2}\right]^2 - 10^2\right)} = 7{\cdot}5\text{; then } 7{\cdot}5 + 12{\cdot}5 = 20\text{; hence}$$

$10^2 : 25 \times 7 :: 20 : 35$ the transverse required.

2. Let the ordinate be 10, the greater abscissa 28, and the conjugate 25; required the transverse diameter?

Ans. 35.

PROBLEM IX.

Given the transverse axis, ordinate, and abscissa, to find the conjugate.

RULE. The square root of the product of the two abscissas is to the ordinate, as the transverse axis is to the conjugate.*

1. Let the transverse axis be 35, the ordinate 10, and the abscissas 28 and 7; required the conjugate?

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and by dividing both sides of the equation by y^2, we get $d^2 - \frac{c^2 x}{y^2} d = -\frac{c^2 x^2}{y^2}$, and by completing the square, we have $d^2 - \frac{c^2 x}{y^2} + \frac{c^4 x^2}{4y^4} = \frac{c^4 x^2}{4y^4} - \frac{c^2 x^2}{y^2}$, then by extracting the square root of both these equal quantities, we get $d - \frac{c^2 x}{2y^2} = \sqrt{\left(\frac{c^4 x^2}{4y^4} - \frac{c^2 x^2}{y^2}\right)}$; hence $d = \frac{c^2 x}{2y^2} \pm \sqrt{\left(\frac{c^4 x^2}{4y^4} - \frac{c^2 x^2}{y^2}\right)} = \frac{cx}{y^2} \times [\frac{1}{2} c \pm \sqrt{(\frac{1}{4} c^2 - y^2)}]$ which is the rule. For $\sqrt{(\frac{1}{4} c^2 - y^2)}$ is the square root of the difference of the squares of the semi-conjugate and ordinate, to which $\frac{1}{2} c$ is added for one factor; then from the nature of proportion $y^2 : cx :: \frac{1}{2} c \pm \sqrt{(\frac{1}{4} c^2 - y^2)} : d = \frac{cx}{y^2} \times [\frac{1}{2} c \pm \sqrt{(\frac{1}{4} c^2 - y^2)}]$.

* *Demonstration.* From the equation of the curve, we have $d^2 : c^2 :: x \times (d - x) : y^2$; the roots of these are proportionals, viz. $\sqrt{[x \times (d - x)]} : y :: d : c$, which is the rule.

$$\sqrt{(28 \times 7)} : 10 :: 35 : \frac{35 \times 10}{\sqrt{(28 \times 7)}} = \frac{35 \times 10}{14} = 25$$

the conjugate.

2. Let the transverse diameter be 120, the ordinate 16, and the abscissas 24 and 96; required the conjugate?

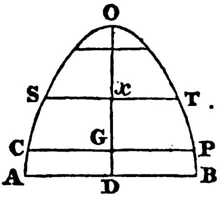
Ans. 40.

OF THE PARABOLA.

PROBLEM X.

Given the base and height of a parabola, to find its area.

Note. Any double ordinate A B or C P of a parabola may be called its base, and the axis O D, or any part of it, as O G its height. Hence it appears that any double ordinate may be its base, and its height the abscissa to that double ordinate.



RULE. Multiply the base by the height, and $\frac{2}{3}$ of the product will be the area*.

1. Required the area of a parabola, whose height is 6 and base 12? $6 \times 12 \times \frac{2}{3} = 48$ the area.

2. What is the area of a parabola, whose base is 24 and 4?

Ans. 64.

3. What is the area of a parabola, whose base is 12 and height 2?

Ans. 16.

* *Demonstration.* This is proved in Proposition XI., Section III., where it is shown that the area of a parabola is equal to $\frac{2}{3}$ of its circumscribing parallelogram. But the base multiplied by the height is the area of the circumscribing parallelogram; then $\frac{2}{3}$ of this area *is the area of* the parabola.

PROBLEM XI.

To find the area of the zone of a parabola, or the space between two parallel double ordinates.

RULE I. When the two double ordinates, their distance, and the altitude of the whole parabola are given; find the area of the whole parabola, and find also the area of the upper segment, their difference will be the area of the zone.

II. When the two double ordinates and their distance are given; to the sum of the squares of the two double ordinates, add their product, divide the sum by the sum of the two double ordinates, multiply the quotient by $\frac{2}{3}$ of the altitude of the zone, and the product will be the altitude of the zone.*

1. Given A B 20, S T 12, and D x 8; what is the area of the zone A S T B, the altitude D O being 12·5?

$(20 \times 12{\cdot}5) \times \frac{2}{3} = 166\frac{2}{3}$ = area of the parabola A B O, and $(12{\cdot}5 - 8) \times 12 = 54$, and $54 \times \frac{2}{3} = 36$; hence $166\frac{2}{3} - 36 = 130\frac{2}{3}$ the area.

III. When the altitude of the whole parabola is not given.

2. Suppose the double ordinate A B = 10, the double

---

* *Demonstration.* By Prop. III. Cor. 1, Sec. III. D O : x O :: A D^2 : Sx^2; and putting A B = D, S T = d, D x = a; D O : x O :: $\frac{1}{4}$D^2 : $\frac{1}{4}d^2$ :: D^2 : d^2; and by division, (17. V.) D O − x O : x O :: D^2 − d^2 : d^2; that is, D x : x O :: D^2 − d^2 : d^2; but D x = a ∴ a : x O :: D^2 − d^2 : d^2. Hence $x\,O = \frac{a\,d^2}{D^2 - d^2}$, and $D\,O = a + \frac{a\,d^2}{D^2 - d^2}$.

But $D \times \left(a + \frac{a\,d^2}{D^2 - d^2}\right) \times \frac{2}{3}$ is equal the area of the whole parabola, and $d \times \frac{a\,d^2}{D^2 - d^2} \times \frac{2}{3}$ is equal the area of the segment S O T; their difference is the area of the zone A S T B, viz.

$$\left(D \times \left(a + \frac{a\,d^2}{D^2 - d^2}\right) \times \tfrac{2}{3}\right) - \left(d \times \frac{a\,d^2}{D^2 - d^2}\right) \times \tfrac{2}{3} = \frac{D^3 - d^3}{D^2 - d^2} \times \tfrac{2}{3}\,a = \frac{D^2 + D\,d + d^2}{D + d} \times \tfrac{2}{3}\,a$$

is the area of the zone A S T B, which is the rule.

ordinate S T $=6$, and their distance D $x=4$; what is the area of the zone A S T B?

$$\frac{10^2+6^2+10\times 6}{10+6}=12\tfrac{1}{4};\ \text{then } 12\tfrac{1}{4}\times 4\times\tfrac{2}{3}=32\tfrac{2}{3}$$

the area as before.

3. Let the double ordinate A B $=30$, C P $=25$, and their distance D G $=6$; required the area of the zone A B P C? *Ans.* $27\frac{19}{33}$.

PROBLEM XII.

To find the length of the curve, or arc of a parabola, cut off by a double ordinate to the axis.

RULE I. Divide the ordinate by the parameter, and call the quotient q.

Add 1 to the quotient q, and call the root of the sum s.

To the product of q and s, add the hyperbolic logarithm of their sum, then the last sum multiplied by half the parameter will give the length of the whole curve on both sides of the axis.

Putting c for the curve, q for the quotient of the double ordinate divided by the parameter, s for $\sqrt{(1+q^2}$, and a for half the parameter; then

$$c=a\times(q\,s+\text{hyp. log. of } q+s).*$$

---

* *Demonstration.* Let v = any curve beginning at the vertex O, y = the ordinate to the axis is at the extremity of the curve, x = its abscissa, $a=\frac{1}{2}$ the parameter of the axis.

From the property of the curve as shown in Prop. III. Cor. 1, Sec. III. its equation is $2\,a\,x=y^2$; the fluxion of these quantities will be equal, hence $2\,a\,\dot{x}=2\,y\,\dot{y}$, and dividing both by $2\,a$, $\dot{x}=\frac{y\,\dot{y}}{a}$; hence $\dot{x}^2=\frac{y^2\,\dot{y}^2}{a^2}\therefore\ \dot{v}=\sqrt{(\dot{y}^2+\dot{x}^2)}$; but $\dot{x}^2=\frac{y^2\,\dot{y}^2}{a^2}\therefore\ \dot{v}=\sqrt{(\dot{y}^2+\frac{y^2\,\dot{y}^2}{a^2})}=\dot{y}\,\frac{\sqrt{(a^2+y^2)}}{a}$. The fluents of these will be $v=y\,\frac{\sqrt{(a^2+y^2)}}{2\,a}$

Note. The common logarithm of any number multiplied by 2·302585093 gives the hyperbolic logarithm of the same number.

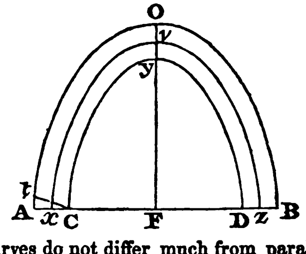
1. What is the length of the curve of a parabola, cut off by a double ordinate to the axis, whose length is 12, the abscissa being 2 ?

$x = 2$,† and $y = 6$; then $a = \frac{y^2}{2x} = \frac{36}{4} = 9$, and $q \frac{y}{a} = \frac{6}{9} = \frac{2}{3}$, also $s = \sqrt{(1 + q^2)} = \sqrt{(1 + \frac{4}{9})} = \sqrt{(\frac{13}{9})} = \frac{1}{3}\sqrt{(13)} = 1{\cdot}2018504 = s$. Hence $\frac{2}{3} + 1{\cdot}2018504 = 1{\cdot}868517$, whose common logarithm is

---

$+ \frac{1}{2} a \times$ hyp. log. of $y + \frac{\sqrt{(a^2 + y^2)}}{a} = \frac{1}{2} a q \sqrt{(1 + q^2)} + \frac{1}{2} a \times$ hyp. log. of $[q + \sqrt{(1 + q^2)}]$, putting $q = \frac{y}{a}$; double of this quantity gives the value of the double curve, viz. $2v = c = a q \sqrt{(1 + q^2)} + a \times$ hyp. log. of $[q + v(1 + q^2)] = a \times (q S +$ hyp. log. of $q + S)$, which is the rule.

† The length of the curve $x v y$ may be found pretty accurately thus: Conceive two curves A O B and C y D to drawn equi-distant from the curve whose length is required at a very small distance, say the one hundred millionth part of an unit from it; then it is obvious



that these two curves do not differ much from parabolas. Find the area of the parabola C y D to the ordinate C F and height F y which call S; find also the area of the parabola A O B to the ordinate A F and height F O, which call G; then G — S will give the area of the ring A C y D B O A, which area being divided by the perpendicular distance O y, or the breadth of the ring, will give the length of a curve equal to a mean between the curve A O B and C y D; that is, of $x v z$; hence $x v z = \frac{G - S}{y O}$. In this figure $O y = t C = \frac{2}{100000000}$, the mean breath of the ring, from which A F and C F may be determined, and hence G and S.

·271497, which being multiplied by 2·302585093, produces ·6251449 for its hyperbolic logarithm; and also $\frac{2}{3} \times$ 1·2018504 = ·8012336; the sum of these two is 1·4263785, therefore $9 \times 1{\cdot}4263785 = 12{\cdot}8374065$, is the length of the curve required.

RULE II. Put y equal to the ordinate, and q equal the quotient arising from the division of the double ordinate by the parameter, or from the division of double the abscissa by the ordinate; then the length of the double curve will be expressed by the infinite series

$$2y \times \left(1 + \frac{q^2}{2{\cdot}3} - \frac{q^4}{2{\cdot}4{\cdot}5} + \frac{3\,q^6}{2{\cdot}4{\cdot}6{\cdot}7}, \text{ \&c.*}\right)$$

Note. This series will converge no longer than till $q = 1$. For when q is greater than 1, the series will diverge.

Let the last example be resumed, in which the abscissa is 2, and the ordinate 6.

Hence $2 \times 2 \div 6 = \frac{2}{3} = q$; then employing $\frac{2}{3}$ instead of q in the last series, we get

$$12 \times \left(1 + \frac{(\frac{2}{3})^2}{2{\cdot}3} - \frac{(\frac{2}{3})^4}{2{\cdot}4{\cdot}5} + 3 \times \frac{(\frac{2}{3})^6}{2{\cdot}4{\cdot}6{\cdot}7}\right) = 12{\cdot}837 \text{ the length}$$

of the curve as before.

RULE III. To the square of the ordinate, add $\frac{4}{3}$ of the square of the abscissa, and the root of the sum will be the length of the single curve, the double of which will be the length of the double curve nearly.†

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* *Demonstration.* The fluxion of the curve in the last rule is $\dot{v} = \dot{y}\sqrt{(1 + \frac{y^2}{a^2})}$ for $\dot{v} = \sqrt{\frac{(\dot{y}^2 + y^2\dot{y}^4)}{a^2}} = \dot{y}\sqrt{(1 + \frac{y^2}{a})}$ = by extracting the square root $\dot{y} \times (1 + \frac{y^2}{2a^2} - \frac{y^4}{2{\cdot}4\,a^4} + \frac{3\,y^6}{2{\cdot}4{\cdot}6\,a^6}$, &c.); then by finding the fluents, and putting $q = \frac{y}{a}$, we get $v = y \times (1 + \frac{q^2}{2{\cdot}3} - \frac{q^4}{2{\cdot}4{\cdot}5} + \frac{3\,q^6}{2{\cdot}4{\cdot}6{\cdot}7}$, &c.) = the length of one side of the curve, the double of which gives the entire length on both sides, viz. $C = 2v = 2y \times (1 + \frac{q^2}{2{\cdot}3} - \frac{q^4}{2{\cdot}4{\cdot}5} + \frac{3\,q^6}{2{\cdot}4{\cdot}6{\cdot}7}$, &c.)

† *Demonstration.* From what has been said in the last rule $v = y$

Note. The two first rules are not recommended in practice.—The practical application of this is much simpler, and is therefore to be employed in peference to either.

Retaining the same example, in which $x = 2$, and $y = 6$, we shall get $v = \sqrt{(y^2 + \frac{4}{3} x^2)} = \sqrt{(36 + \frac{16}{3})} = 6{\cdot}1291$, and $C = 12{\cdot}8582$, nearly.

2. Required the length of the parabolic curve, whose abscissa is 3, and ordinate 8? *Ans.* 17·384.

PROBLEM XIII.

Given any two abscissas and the ordinate to one of them, to find the corresponding ordinate to the second abscissa.

RULE. Say, as the abscissa, whose ordinate is given, is to the square of the given ordinate, so is the other given abscissa to the square of its corresponding ordinate.*

1. If the abscissa x O = 10, and the ordinate x S = 8, what is the ordinate A D, whose abscissa D O is 20?

$x\text{O} : x\text{S}^2 :: \text{DO} : \text{AD}^2$, viz. $10 : 64 :: 20 : 128$, the square root of which is 11·313, &c., = A D.

2. If 6 be the ordinate corresponding to the abscissa 9, required the ordinate corresponding to the abscissa 16? *Ans.* 8.

PROBLEM XIV.

Given two ordinates, and the abscissa corresponding to one of them, to find the abscissa corresponding to the other.

RULE. Say, as the square of the ordinate whose ab-

$\times \left(1 + \frac{1}{2\cdot 3} q^2 - \frac{1}{2\cdot 4\cdot 5} q^4 + \frac{3}{2\cdot 4\cdot 6\cdot 7} q^6, \text{\&c.,}\right)$ and as $\sqrt{(1 + \frac{1}{3} q^2)} = 1 + \frac{1}{2\cdot 3} q^2 - \frac{1}{2\cdot 4\cdot 9} q^4 + \frac{3}{2\cdot 4\cdot 6\cdot 27} q^6$, &c.; hence $\frac{c}{y} - \sqrt{(1 + \frac{1}{3} q^2)} = - \frac{1}{2\cdot 5\cdot 9} q^4 + \frac{5}{2\cdot 7\cdot 27} q^6$, &c.; then if we suppose q not greater than 1, and reject the series, $v = y\sqrt{(1 + \frac{1}{3} q^2)} = \sqrt{(y^2 + \frac{4}{3} x^2)}$ nearly; $\because C = 2v = 2\sqrt{(y^2 + \frac{4}{3} x^2)}$ nearly.

* This is evident from Proposition III. Cor. 1. Section III.

scissa is given, is to the given abscissa, so is the square of the other ordinate to its corresponding abscissa.*

1. Given $S x = 6$, $x O = 9$, and $A D = 8$; required the abscissa O D? $36 : 9 :: 64 : 16 = O D$.

2. Given $S x = 8$, $x O = 10$, and $A D = 9$; required O D? *Ans.* 12·656.

PROBLEM XV.

Given two ordinates perpendicular to the axis and their distance, to find the corresponding abscissas.

RULE. Say, as the difference of the squares of the ordinates is to their distance, so is the square of either of them to the corresponding abscissa.†

1. Given $S x = 6$, $A D = 8$, and $x D = 7$; required the abscissas?

$$(64 - 36) : 7 :: 64$$
$$28 : 7 :: 64 : 16 = O D, \text{ and}$$
$$28 : 7 :: 36 : 9 = O x.$$

2. Given $S x = 3$, $A D = 4$, and $x D = 2$; required the abscissas? *Ans.* $4\frac{4}{7}$ and $2\frac{4}{7}$.

OF THE HYPERBOLA.

PROBLEM XVI.

Given the transverse and conjugate diameters, as also any abscissa, to find the corresponding ordinate.

RULE. The transverse is to the conjugate, so is the mean proportional between the abscissas, to the ordinate.‡

* This rule is derived from Proposition III. Cor. 1. Section III.

† *Demonstration.* From the property of the parabola, $D O : x O :: A D^2 : S x^2$; then by division, $D O - x O (= D x) : x O :: A D^2 - S x^2 : S x^2$ that is, $A D^2 - S x^2 : S x^2 :: D x : x O$. Also, $A D^2 - S x^2 :: A D^2 :: D x : D O$.

‡ *Demonstration.* It is proved in Proposition V. Cor. 1, Section III.

1. If the transverse be 24, the conjugate 21, and the less abscissa 8; required the ordinate?

$24 : 21 :: \sqrt{(32 \times 8)} : \frac{21\sqrt{(32 \times 8)}}{24} = 14$ the ordinate.

2. If the transverse axis of an hyperbola be 120, the less abscissa 40, the conjugate 72; required the ordinate.

Ans. 48.

3. The transverse axis being 60, the conjugate 36, and the less abscissa 20, what is the ordinate? *Ans.* 24.

PROBLEM XVII.

Given the transverse, conjugate, and ordinate, to find the abscissa.

RULE. To the square of half the conjugate, add the square of the ordinate, and extract the square root of the sum. Then say,

As the conjugate is to the transverse, so is that square root to half the sum of the abscissas.

Then to this half sum, add half the transverse, for the greater abscissa; and from the half sum take half the transverse for the less abscissa.*

---

that the square of half the transverse is to the square of half the conjugate, as the rectangle of the abscissa by the sum of the abscissa and transverse (which is the abscissa relatively to the opposite curve,) to the square of the ordinate; therefore $t^2 : c^2 :: x \times (t + x) : y^2$, and $t : c :: \sqrt{(x \times (t + x))} : y$, which is the rule; t, c, x, a, y, being the transverse, conjugate, abscissa, and ordinate.

Note. The less abscissa added to the transverse is equal to the greater.

* *Demonstration.* Retaining the same letters as in the last demonstration, $\left(\frac{c}{2}\right)^2 : \left(\frac{t}{2}\right)^2 :: y^2 + \left(\frac{c}{2}\right)^2 : \left(\frac{t}{2} + x\right)^2$, (Prop. V. Cor. 4. Sec. III.) then $c : t :: \sqrt{\left(y^2 + \left(\frac{c}{2}\right)^2\right)} : \frac{t}{2} + x =$ half the abscissas; then $\frac{t}{2} + x + \frac{t}{2} = t + x =$ greater abscissa, and $\frac{t}{2} + x - \frac{t}{2} = x\ t$

$\frac{\sqrt{\left(\left(\frac{c}{2}\right)^2 + y^2\right)}}{c} \pm \frac{t}{2} =$ the greater or less abscissa, which is the rule.

1. If the transverse be 24, and the conjugate 21; required the abscissas to the ordinate 14?

$$\begin{array}{ll} 10{\cdot}5 = \tfrac{1}{2} \text{ conjugate} & 14 = \text{ordinate} \\ 10{\cdot}5 & 14 \\ \hline 110{\cdot}25 & 196 \\ 196 & \\ \hline \end{array}$$

306·25 the square root of which is 17·5; then 21 : 24 : : 17·5 : 20 = half sum, 20 + 12 = 32 the greater abscissa, and 20 — 12 = 8 the less abscissa.

2. The transverse is 120, the ordinate 48, and the conjugate 72; required the abscissas? *Ans.* 40 and 160.

PROBLEM XVIII.

Given the conjugate, ordinate, and abscissas, to find the transverse.

RULE. To or from the root of the sum of the squares of the ordinate and semi-conjugate, add or subtract the semi-conjugate, according as the less or greater abscissa, is used; then as the square of the ordinate, is to the product of the abscissa and conjugate; so is the sum or difference, above found, to the transverse.*

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* *Demonstration.* From Proposition V. Cor. 1. Section III., we have $\left(\frac{t}{2}\right)^2 : \left(\frac{c}{2}\right)^2 :: x \times (t+x) : y^2$; hence $t^2 : c^2 :: xt + x^2 : y^2$, then $y^2 t^2 = c^2 x t + c^2 x^2$; and $y^2 t^2 - c^2 x t = c^2 x^2$; divide by y^2, and we get $t^2 - \frac{c^2 x}{y^2} \times t = \frac{c^2 x^2}{y^2}$, complete the square, and then $t^2 - \frac{c^2 x}{y^2} \times t + \frac{c^4 x^2}{4 y^4} = \frac{c^2 x^2}{y^2} + \frac{c^4 x^2}{4 y^4} = \frac{4 c^2 x^2 y^2 + c^4 x^2}{4 y^4}$, and $t - \frac{c^2 x}{2 y^2} = \sqrt{\left(\frac{4 c^2 x^2 y^2 + c^4 x^2}{4 y^4}\right)} = \sqrt{\left[\frac{(4 y^2 + c^2) \times c^2 x^2}{4 y^4}\right]} = \frac{c x}{2 y^2}$

1. Let the conjugate be 21, the less abscissa 8, and its ordinate 14; required the transverse?

$$\frac{cx \times \sqrt{(y^2 + \frac{1}{4}c^2) + \frac{c}{2}}}{y^2} = \frac{21 \times 8 \times \sqrt{(14^2 + \frac{21^2}{4} + 10\frac{1}{2})}}{14^2}$$

$= 3 \times (\sqrt{(3^2 + 4^2)} + 3) = 3 \times (5 + 3) = 24$ the transverse.

2. The conjugate axis is 72, the less abscissa 40, the ordinate 48; required the transverse? *Ans.* 120.

3. The conjugate is 36, the less abscissa 20, and its ordinate 24; required the transverse? *Ans.* 60.

PROBLEM XIX.

Given the abscissa, ordinate, and transverse diameter; to find the conjugate.

RULE. As the mean proportional between the abscissas, is to the ordinate, so is the transverse to its conjugate.*

1. What is the conjugate to the transverse 24, the less abscissa being 8, and its ordinate 14?

$$\frac{ty}{\sqrt{((t+x) \times x)}} = \frac{24 \times 14}{\sqrt{(32 \times 8)}} = 21 \text{ the conjugate.}$$

2. The transverse diameter is 60, the ordinate 24, and the less abscissa 20; what is the conjugate? *Ans.* 36.

$\sqrt{(4y^2 + c^2)}t = \frac{c^2 x}{2y^2} + \frac{cx}{2y^2}\sqrt{(4y^2 + c^2)} = \frac{cx}{2y^2} \times \sqrt{(4y^2 + c^2)}$

$+ c = \frac{cx \times \sqrt{(y^2 + \frac{1}{4}c^2) + \frac{c}{2}}}{y^2}$, which is the rule.

* *Demonstration.* By the last, we have $t^2 : c^2 :: (t+x) \times x : y^2$; then the roots of these are proportionals; viz. $t : c :: \sqrt{[(t+x) \times x]} : y$; and then $\sqrt{[(t+x) \times x]} : y :: t : c = \frac{ty}{\sqrt{[(t+x) \times x]}}$, which is the rule.

PROBLEM XX.

Given any two abscissas, X, x, *and their ordinates,* Y, y; *to find the transverse to which they belong.*

RULE. Multiply each abscissa by the square of the ordinate belonging to the other; multiply also the square of each abscissa by the square of the other's ordinate; then divide the difference of the latter products by the difference of the former; and the quotient will be the transverse diameter to which the ordinates belong.*

1. If two abscissas be 1 and 8, and their corresponding ordinates $4\frac{3}{8}$ and 14, required the transverse to which they belong?

Here $\dfrac{8^2 \times 4\frac{3}{8} \times 4\frac{3}{8} - 1^2 \times 14^2}{1 \times 14^2 - 8 \times 4\frac{3}{8} \times 4\frac{3}{8}} = \dfrac{35 \times 35 - 14 \times 14}{14 \times 14 - 35 \times 4\frac{3}{8}}$

$= \dfrac{5 \times 5 - 2 \times 2}{2 \times 2 - 5 \times \frac{3}{8}} = \dfrac{21 \times 8}{7} = 24$ the transverse.

PROBLEM XXI.

To find the area of a hyperbola by means of equi-distant ordinates.

Let A N be divided into any even number of equal parts, A C, C E, E G, &c., and let perpendicular ordinates A B, C D, E F, &c. be erected, and let these ordinates be terminated by any hyperbolic curve B D F, &c.; and let A = A B + N O, B = C D + G H + L M, &c., and C = E F + I K, &c.; then the common distance A C, of the ordinates, being multiplied by the sum arising from the addition

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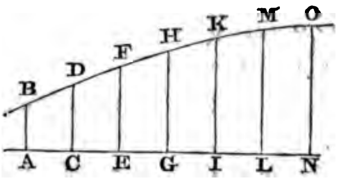
* *Demonstration.* Let X and x be the abscissas, and Y and y the corresponding ordinates; then by Proposition V. Cor. 3. Section III., we have $t\,x + x^2 : t\,X + X^2 :: y^2 : Y^2$; hence $t\,X\,y^2 + X^2\,y^2 = t\,x\,Y^2 + x^2\,Y^2$, then $t\,X\,y^2 - t\,x\,Y^2 = x^2\,Y^2 - X^2\,y^2$, and $t = \dfrac{x^2\,Y^2 - X^2\,y^2}{X\,y^2 - x\,Y^2}$, or $t = \dfrac{X^2\,y^2 - x\,Y^2}{x\,Y^2 - X\,y^2}$, which is the rule.



of A, 4 B, and 2 C, one-third of the product will be the area very nearly. That is, $\frac{A + 4B + 2C}{3} \times D =$ the area, putting $D = AC$.

Conceive a parabolic curve to pass through the first three points B D F, which will very nearly coincide with the hyperbolic curve, when the ordinates are taken very near each other; and therefore the area of the hyperbolic space A B F E will be very nearly equal to the parabolic space, their boundaries nearly coinciding. Let the abscissa be called x, and the ordinate y; then from the nature of the parabola we have $\frac{c^2}{t^2} \times (x^2 + t\,x) = y^2$, and $y = \frac{c}{t} \times \sqrt{(x^2 + t\,x)}$. By extracting the square root of $\sqrt{(x^2 + t\,x,)}$ which will be an infinite series, and multiplying by $\frac{c}{t}$, we shall have y. Let the infinite series multiplied by $\frac{c}{t}$ be represented by $A + Bx + Cx^2 + Dx^3$, &c., which is obviously a general expression for the ordinate y; and if we consider x to be composed of an infinite number of points, beginning with a cypher, the several values of it, as it increases from 0, may be represented by 0, 1, 2, 3, &c. Now as all the ordinates (which are supposed to be extremely near each other,) corresponding to the abscissas 0, 1, 2, 3, &c., make up the area of the space A B F E, let x be interpreted by 0, 1, 2, 3, &c., and we shall get

$A + B \times 0 + C \times 0^2$ $= A$ or AB the ordinate, when $x = 0$.
$A + B \times 1 + C \times 1^2$
$A + B \times 2 + C \times 2^2$

. . . .

$A + B \times \frac{x}{2} + C \times \frac{x^2}{4}$ $=$ middle ordinate DC.

. . . .

$A + B \times x + C \times x^2$ $=$ third ordinate EF.

The sum of all these is obviously the area of the space $ABFE$; that is, $Ax + \frac{Bx^2}{2} + \frac{Cx^3}{3} =$ area $=$

$$\frac{6Ax + 3Bx^2 + 2Cx^3}{6} = \frac{x}{6} \times (6A + 3Bx + 2Cx^2)$$

From the middle equation, we have

$4A + 2Bx + Cx^2 = 4DC$, and when $x = 0$, we get
$A = AB$, when $x = AE$, we get
$A + Bx + Cx^2 = EF$, add these together, and
$6A + 3Bx + 2Cx^2 = 4DC + AB + EF$. But

$(6A + 3Bx + 2Cx^2) \times \frac{x}{6} =$ area of the curve $\therefore$

$(AB + 4DC + EF) \times \frac{x}{6} =$ area of the curve; or

$(AB + 4DC + EF) \times \frac{D}{3}$; for $D = x \div 2$. Pursuing the same mode $(EF + 4GH + IK) \times \frac{D}{3} =$ area of the hyperbolic space $EFKI$; and $(IK + 4LM + NO) \times \frac{D}{3} =$ area of $IKON$, &c., consequently the sum of all these areas will give the entire hyperbolic space $ABON$ very nearly; that is, $(AB + 4CD + 2EF + 4GH + 2IK + 4LM + NO) \times \frac{D}{3} = (A + 4B + 2C) \times \frac{D}{3}$.

1. *Given eleven ordinates to an hyperbola, between the asymptotes, to find the area.* Let us suppose the first value

of x to be nothing, and the last 1, therefore the common distance of the ordinates will be $\frac{1}{10}$, and from the nature of the curve we shall have the ordinates $\frac{1}{1+0}$, $\frac{1}{1+\cdot1}$, $\frac{1}{1+\cdot2}$, &c.

Or, $\frac{10}{10}$, $\frac{10}{11}$, $\frac{10}{12}$, $\frac{10}{13}$, $\frac{10}{14}$, $\frac{10}{15}$, $\frac{10}{16}$, $\frac{10}{17}$, $\frac{10}{18}$, $\frac{10}{19}$, $\frac{10}{20}$.

Here $A = \frac{10}{10} + \frac{10}{20} = 1\frac{1}{2} = 1{\cdot}5$.

$B = \frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19} = 3{\cdot}4595393$.

$C = \frac{10}{12} + \frac{10}{14} + \frac{10}{16} + \frac{10}{18} = 2{\cdot}7281745\frac{5}{6}$.

$D = \cdot1$. Then $(A + 4\,B + 2\,C) \times \frac{D}{3} =$ $\cdot69315021$ the area required.

This formula will answer for finding the contents of all solids, by using the sections perpendicular to the axis. The greater the number of ordinates employed, the more accurate the result; but in real practice three or five are in most cases sufficient.

PROBLEM XXII.

To find the length of any arc of an hyperbola, beginning at the vertex.

RULE I. To 19 times the square of the transverse, add 21 times the square of the conjugate; also to 9 times the square of the transverse add, as before, 21 times the square of the conjugate, and multiply each of these sums by the abscissa.

II. To each of these two products, thus found, add 15 times the product of the transverse and the square of the conjugate.

III. Then as the less of these results is to the greater, so is the ordinate to the length of the curve nearly.*

* Let a = semi-transverse, b = semi-conjugate, x = the abscissa, y = ordinate; then $y^2 = \frac{b^2}{a^2} \cdot (2\,a\,x + x^2)$ hence $x^2 + 2\,a\,x = \frac{a^2y^2}{b^2}$,

1. In the hyperbola B A C, the transverse diameter is 80, the conjugate 60, the ordinate B D 10, and the abscissa A D 2; required the length of the arc B A C ?* (Fig. p. 150.)

Here $2(19\times80^2+21\times60^2)=2(121600+75600)=314400$.

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and $x=\frac{a}{b}\cdot(b^2+y^2)^{\frac{1}{2}}-a$, and $dx=\frac{a y\, dy}{b\,(b^2 y^2)^{\frac{1}{2}}}$. And $dz=$

$$\left\{dy^2+\frac{a^2y^2dy^2}{b^2(b^2+y^2)}\right\}^{\frac{1}{2}}=dy\cdot\left\{1+\frac{a^2y^2}{b^4+b^2y^2}\right\}^{\frac{1}{2}}=$$

$$dy\left\{1+\frac{a^2y^2}{b^4}-\frac{a^2y^4}{b^6}+\frac{a^2y^6}{b^8}-\frac{a^2y^8}{b^{10}}+\&c.\right\}^{\frac{1}{2}}=$$

$$dy\left\{1+\frac{a^2y^2}{2b^4}-\left(\frac{a^2}{2b^6}+\frac{a^4}{8b^8}\right)y^4+\left(\frac{a^2}{2b^8}+\frac{a^4}{4b^{10}}+\frac{a^6}{16b^{12}}\right)y^6-\&c.\right\}$$

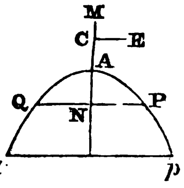
$$=dy\left\{1+\frac{a^2y^2}{2b^4}-\frac{a^4+4a^2b^2}{8b^8}\cdot y^4+\frac{a^6+4a^4b^2+8a^2b^4}{16b^{12}}\cdot y^6-\&c.\right\}$$

$$\therefore z=y\left\{1+\frac{a^2}{6b^4}\cdot y^2-\frac{a^4+4a^2b^2}{40b^8}\cdot y^4+\frac{a^6+4a^4b^2+8a^2b^4}{112b^{12}}\cdot y^6-\&c.\right\}=\text{the arc A P.}$$

But the rule expressed algebraically is

$\frac{15ab^2+(19a^2+21b^2)\cdot x}{15ab^2+(9a^2+21b^2)\cdot x}\times y$, which if actually divided, the quotient will be found to differ but little from the preceding series which expresses the true value of the arcs, therefore the rule approximates the truth.

Note. It is to be observed that $x=\frac{a}{b}\cdot(b^2+y^2)^{\frac{1}{2}}-a$.



* The length of an arc of an hyperbola may be found thus: Conceive two curves to be drawn equi-distant from Q A P at an extremely small distance from it, as in the parabola; find the area of these two new figures taken as hyperbolas, from which they do not materially differ by reason of their nearly coinciding with the hyperbola Q A P; then the difference between these two areas divided by the perpendicular distance between them will give the length of the curve Q A P nearly.

Hence, any curve may be rectified in this way, when its area can be found from certain dimensions.

And $2(9 \times 80^2 + 21 \times 60^2) = 2(57600 + 75600) = 266400$.

Whence $15 \times 80 \times 60^2 + 314400 = 4320000 + 314400 = 4634400$.

And $15 \times 80 \times 60^2 + 266400 = 4320000 + 266400 = 4586400$.

Then $4586400 : 4634400 :: 10 : \frac{46344000}{4586400} = 10{\cdot}1046 = AB$.

Hence $ABC = 10{\cdot}1046 \times 2 = 20{\cdot}2092$.

2. In the hyperbola B A C, the transverse diameter is 80, the conjugate 60, the ordinate B D 10, and the abscissa A D 2·1637; required the length of the arc A B?

Ans. 10·3005.

3. Required the area of the hyperbola to the abscissa 25, the axes being 50 and 30? *Ans.* 805·0909.

PROBLEM XXIII.

Given the transverse axis of a hyperbola, the conjugate, and the abscissa, to find the area.

RULE I. To the product of the transverse and abscissa, add $\frac{5}{7}$ of the square of the abscissa, and multiply the square root of the sum by 21.

II. Add 4 times the square root of the product of the transverse and abscissa, to the preceding product, and divide the sum by 75.

III. Divide 4 times the product of the conjugate and abscissa by the transverse; this quotient, multiplied by the former quotient, will give the area of the hyperbola nearly.*

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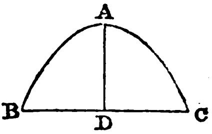
* *Demonstration.* Retaining the same notation, the equation of the hyperbola is $y = \frac{b}{a}(2ax + x^2)^{\frac{1}{2}}$. Hence $y\,dx = \frac{b}{a}(2ax + x^2)^{\frac{1}{2}}$.

1. In the hyperbola B A C, the transverse axis is 30, the conjugate 18, and the abscissa A D is 10; what is the area ?



Here $21\sqrt{(30 \times 10 + \frac{5}{7} \times 10^2)} = 21\sqrt{(300 + 71{\cdot}42857)} = 21\sqrt{(371{\cdot}42857)} = 21 \times 19{\cdot}272 = 404{\cdot}712$;

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dx. Now in order to make the series expressing the area converge, let $w = \frac{x}{2a+x}$, from which $x = \frac{2aw}{1-w}$. And

$$y\,dx = \frac{b}{a}\left\{\frac{4a^2w}{1-w} + \frac{4a^2w^2}{(1-w)^2}\right\}^{\frac{1}{2}} \cdot \frac{2a\,dw}{(1-w)^2} = \frac{4aw^{\frac{1}{2}}dw}{(1-w)^3} =$$

$2abw^{\frac{1}{2}}dw\left\{1{\cdot}2 + 2{\cdot}3w + 3{\cdot}4w^2 + 4{\cdot}5w^3 + \&c.\right\}$ by actual division; therefore $\int y\,dx = 4abw^{\frac{3}{2}}\left\{\frac{1{\cdot}2}{3} + \frac{2{\cdot}3}{5}w + \frac{3{\cdot}4}{7}w^2 + \&c.\right\}$

But as $w = \frac{x}{2a+x}$ is a proper fraction, the powers of w in this series converge, while the co-efficients diverge; therefore in order to make the co-efficients as well as the powers of w to converge, we multiply the series last obtained by $(1-w)$, also divide the factor $4abw^{\frac{3}{2}}$ by the same quantity, and we get

$$\int y\,dx = \frac{8abw^{\frac{3}{2}}}{1-w}\left\{\frac{1{\cdot}1}{1{\cdot}3} + \frac{2{\cdot}2}{3{\cdot}5}w + \frac{3{\cdot}3}{5{\cdot}7}w^2 + \frac{4{\cdot}4}{7{\cdot}9}w^3 + \&c.\right\}$$

By repeating the operation the value of the series is not altered; hence

$\int y\,dx = \frac{8abw^{\frac{3}{2}}}{(1-w)^2}\left\{\frac{1}{3} - \frac{1}{1{\cdot}3{\cdot}5}w - \frac{1}{3{\cdot}5{\cdot}7}w^2 - \frac{1}{5{\cdot}7{\cdot}9}w^3 - \&c.\right\}$ but

$y = \frac{b}{a}(2ax + x^2)^{\frac{1}{2}} = \frac{2bw^{\frac{1}{2}}}{1-w}$, and $x = \frac{2aw}{1-w}$ $\therefore 2xy = \frac{8abw^{\frac{3}{2}}}{(1-w)^2}$;

therefore,

$\int y\,dx = 2xy\left\{\frac{1}{3} - \frac{1}{1{\cdot}3{\cdot}5}w - \frac{1}{3{\cdot}5{\cdot}7}w^2 - \frac{1}{5{\cdot}7{\cdot}9}w^3 - \&c.\right\} =$ area *APN*. *Hence* area A P Q $=$

And $\frac{4\sqrt{(30 \times 10)} + 404{\cdot}712}{75} = \frac{4 \times 17{\cdot}3205 + 403{\cdot}712}{75}$

$= \frac{69{\cdot}282 + 404{\cdot}712}{75} = \frac{473{\cdot}994}{75} = 6{\cdot}3199.$

Whence $\frac{18 \times 10 \times 4}{30} \times 6{\cdot}3199 = 24 \times 6{\cdot}3199 =$ 151·6876 the area required.

2. What is the area of an hyperbola whose abscissa is 25, and transverse and conjugate are 50 and 30?

Ans. 805·0909.

3. The transverse axis is 100, the conjugate 60, and abscissa 50; required the area? *Ans.* 322·3633584.

$$4\,x\,y\left\{\frac{1}{3} - \frac{1}{1{\cdot}3{\cdot}5}\,w - \frac{1}{3{\cdot}5{\cdot}7}\,w^2 - \&c.\right\} =$$

$$4\,x\,y\left\{\frac{1}{3} - \frac{1}{1{\cdot}3{\cdot}5}\cdot\frac{x}{2a+x} - \frac{1}{3{\cdot}5{\cdot}7}\cdot\frac{x^2}{(2a+x)^2} - \&c.\right\}$$ But

$y = \frac{b}{a}(2\,a\,x + x^2)^{\frac{1}{2}}$. And the rule is

$\frac{21\,(2\,a\,x + \frac{5}{7}\,x^2)^{\frac{1}{2}} + 4\sqrt{(a\,x)}}{75} \times \frac{4\,b\,x}{a}$, which being expanded will produce a series nearly equal to the true expression found for the area. (See Fig. page 148.)

MENSURATION OF SOLIDS.

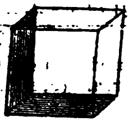
SECTION V.

DEFINITIONS.

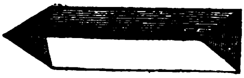
1. A *solid* is that which has length, breadth, and thickness.

2. The least measure is a cubic inch, and all solid bodies are measured by cubes, whose sides are inches, feet, yards, &c.

3. The solid contents of any body is the number of cubic inches, feet, yards, &c., it contains.

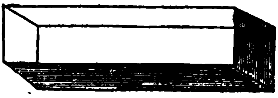


4. A *cube* is a solid, having six equal sides at right angles to one another.



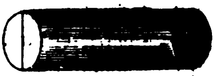
5. A *prism* is a solid, whose ends are any plane figure, which are parallel, equal and similar. Its sides are parallelograms.

It is called a triangular prism, when its ends are triangles; a square prism, when its ends are squares; a pentagonal prism, when its ends are pentagons; and so on.

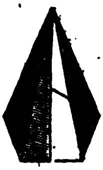


6. A *paralellopipedon* is a solid having six rectangular sides, every opposite pair of *which are* equal and parallel.

7. A *cylinder* is a round prism, having circular ends.



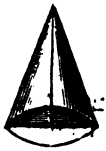
8. A *pyramid* is a solid, having any plane figure for its base; its sides are triangles meeting in a point, called the vertex.



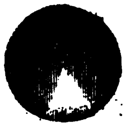
Pyramids have their names from their bases, like the prism.

When the base is a triangle, the solid is called a triangular pyramid; when the base is a square, it is called a square pyramid; and so on.

9. A *cone* is a round pyramid, having a circle for its base.



10. A *sphere* is a round solid, which may be conceived to be formed by the revolution of a semi-circle about its diameter which remains fixed.



11. The axis of a solid is a line joining the middle of both ends.

12. When the axis is perpendicular to the base, the solid is called a right prism or pyramid, otherwise it is oblique.

13. The height or altitude of a solid, is a line drawn from its vertex, perpendicular to its base, and is equal to the axis of a right prism or pyramid; but in an oblique one, the altitude is the perpendicular of a right-angled triangle, whose hypothenuse is the axis.

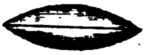
14. When the base is a regular figure, it is called a regular prism or pyramid; but when the base is an irregular figure, the solid on it is called irregular.

15. The segment of any solid, is a part cut off from the top by a plane parallel to its base.

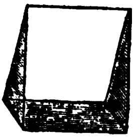
16. A frustrum is the part remaining at the bottom, after the segment is cut off.

17. A zone of a sphere is a part intercepted between two planes, which are parallel to each other.

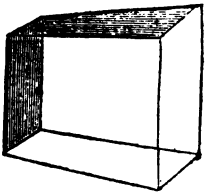
18. A *circular spindle* is a solid generated by the revolution of a segment of a circle about its chord, which remains fixed.



19. A *wedge* is a solid, having a rectangular base, and two of its opposite sides meeting in an edge.



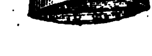
20. A *prismoid* is a solid, having for its two ends two right-angled parallelograms, parallel to each other, and its upright sides are four trapezoids.



21. A *spheroid* is a solid, generated by the rotation of a semi-ellipsis about one of its axes, which remains fixed.

When the ellipsis revolves round the transverse axis, the figure is called a prolate, or oblong speroid; but when the ellipsis revolves round the shorter axis, the figure is called an oblate spheroid.

22. An elliptical spindle is a solid, generated by the rotation of a segment of an ellipsis about its chord.



23. A *parabolic conoid*, or paraboloid, is a solid generated by the rotation of a semiparabola about its axis.

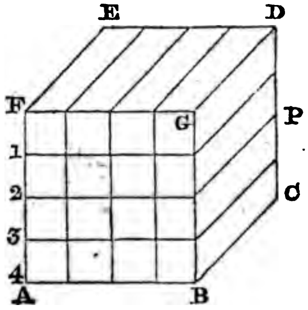
24. An *ungula*, or *hoof*, is a part cut off a solid by a plane oblique to the base.

PROBLEM I.

To find the solidity of a cube.

RULE. Multiply the side of a cube by itself, and that product again by the side, for the solidity required.*

1. If the side of a cube be 4 inches, required its solidity?



* If we conceive a plane to pass through each of the lineal measuring unit, parallel to the ends; and the ends to be similarly divided by planes passing through each lineal measuring unit parallel to the sides; then it is evident that the part cut off will be divided by the plane into as many cubes as there are squares in each end.

But the magnitude of the whole prism, or of any other of an equal base, is to the magnitude of the part whose height is the lineal measuring unit, as the height of the whole prism is to 1; therefore the solidity of the whole is equal to that of the part repeated as often as there are lineal measuring units in the height; that is, equal to the base multiplied by the height.

This rule is true for oblique prisms, as is evident, by conceiving a right and an oblique prism of equal bases and heights, to be made up of an infinite number of plates infinitely thin, all parallel to the base; when the prisms are of the same height, the right and oblique prism will require the same number of such plates, and therefore both must be equal to each other, as they require the same number of equal plates to constitute them.

Here, $4 \times 4 = 16$, the number of cubes of 1 inch deep in the square E F G D, and as the entire solid consists of four such dimensions, its contents is $16 \times 4 = 64$ cubic inches.

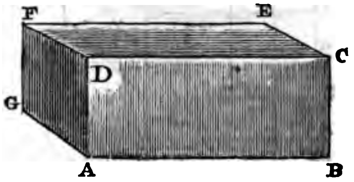
2. What is the solidity of a cubical piece of marble, each side being 5 feet 7 inches? *Ans.* 174 feet, nearly.

3. A cellar is to be dug, whose length, breadth, and depth, are each 12 feet 3 inches, required the number of solid feet in it? *Ans.* 1838 feet 3 inches, nearly.

PROBLEM II.

To find the solidity of a parallelopipedon.

RULE. Multiply continually the length, breadth, and depth together, for the solidity.



1. What is the solidity of the parallelopipedon ABCDEFG, the length A B being 10 feet, the breadth A G 4 feet, and thickness A D 5 feet?

$A B \times A G \times A D = 10 \times 4 \times 5 = 200$ feet.

Note. This rule is evident from what has been said in the last.

2. A piece of timber 26 feet long, 10 inches broad, and 8 inches deep; required its solid contents? *Ans.* 2080 feet.

3. A piece of timber is 10 inches square at the ends and 40 feet long; required its contents? *Ans.* 4000 feet.

4. A piece of timber 15 inches square at each end, and 18 feet long, is to be measured, required its contents, and how far from the end must it be cut across, so that the piece cut off may contain 1 solid foot?

Ans. The solidity is 28·125 feet; and 7·68 in length will make one foot.

4. What length of a piece of square timber will make one solid foot, being 2 feet 9 inches deep, and 1 foot 7 inches broad?

Ans. 2·756 inches in length will make one solid foot.

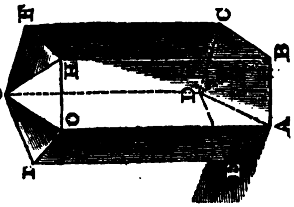
PROBLEM III.

To find the solidity of a prism.

RULE. Multiply the area of the base by the perpendicular height and the product will be the solidity.

Note. The reason of this rule is evident from what was said in Rule I.

1. What is the solidity of a prism, A B C F I E, whose



side C D is a pentagon, each side of which being 3·75, and height 15 feet?

When the side of a pentagon is 1, its area is 1·720477, (Table II.); therefore $1·720477 \times 3·75^2 = 24·1942 =$ the area of the base in square feet; hence $24·1942 \times 15 = 362·913$ solid feet, the content.

2. What is the solidity of a square prism whose length is $5\frac{1}{2}$ feet, and each side of its base $1\frac{1}{3}$ foot?

Ans. $9\frac{7}{9}$ solid feet.

3. What is the solidity of a prism, whose base is an equilateral triangle, each side being 4 feet, and height 10 feet?

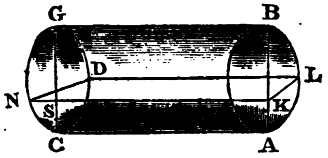
Ans. 69·282 feet.

4. What quantity of water will a prismatic vessel contain, it being a square, each side of which is 3 feet, and height 7 feet? *Ans.* 63 feet.

PROBLEM IV.

To find the solidity of a cylinder.

RULE. Multiply the area of the base by its height, and the product will be the solid content.*



1. What is the capacity of a right cylinder A B G C, whose height, and the circumference of its base, are (each) 20 feet?

First $\frac{20}{3\cdot1416}$ = the diameter, half of which multiplied by half the circumference will give the area of the base, (Prob. XVIII. Sec. II.), that is, $10 \times \frac{10}{3\cdot1416} = \frac{25}{\cdot7854}$ = the area of the end; then $\frac{25}{\cdot7854} \times 20 = 636\cdot61828$ the content.

2. What is the contents of the oblique cylinder A B F E, the circumference of whose base is 20 feet, and altitude A C 20 feet?

As before, the area of the base is $\frac{25}{\cdot7854}$; then $\frac{25}{\cdot7854} \times 20 = 636\cdot61828$, the solid contents, as before.

3. The length of a cylindrical piece of timber is 18 feet,

* The reason of this rule is evident from Rule I.

and its circumference 96 inches; how many solid feet in it? *Ans.* 91·676 feet.

4. Three cubic feet are to be cut off a rolling stone 44 inches in circumference; what distance from the end must the section be made? *Ans.* 33·64 inches.

PROBLEM V.

To find the solidity of the prisms formed by a plane passing parallel to the axis of a cylinder.

RULE. Find by Prob. XXVII. Sec. II. the area of each segment of the base, which multiplied by the height, will give the solidity of each prism.*

1. In the cylinder A B G C, whose diameter is 3, and height 20 feet; let a plane K N pass parallel to the axis, and 1 foot from it; what is the solidity of the two prisms into which the cylinder is divided.—(See the last figure.)

$\frac{SC}{GC} = (\frac{3}{2} - 1) \div 3 = \frac{\frac{1}{2}}{3} = \frac{1}{6} = \cdot 16$ the tabular versed sine, to which in the Table of Circular Segments corresponding to the area is .. .. ·08604117
which taken from .. .. .. ·78539816
leaves the other segment .. .. ·69935699

Then $3^2 = 9 \times \cdot 08604117 = 7\cdot 7437053 =$ seg. D C N.

Also $3^2 = 9 \times \cdot 69935699 = 6\cdot 29421291 =$ seg. D G N.

Hence $20 \times 7\cdot 7437053 = 15\cdot 4874 =$ the slice L K A C N D; and $20 \times 6\cdot 29421699 = 125\cdot 88434 =$ the slice L K B G N D.

Suppose the right cylinder, whose length is 20 feet, and diameter 50 feet, is cut by a plane parallel to, and at the distance of, 21·75 feet from its axis; required the solidity of the smaller slice? *Ans.* 1082·95 feet.

* The reason of this rule is manifest from Prob. XXVII. Sec. II. and from Prob. I. in this Section.

PROBLEM VI.

To find the solidity of a pyramid.

RULE. Multiply the area of the base by the one-third of the height, and the product will be the solidity.*

1. What is the solidity of a square pyramid, each side of its base being 4 feet, and height 12 feet?

$4 \times 4 = 16 =$ the area of the base:

Then $16 \times \frac{12}{3} = 64$ feet, the solidity.

2. Eeach side of the base of a triangular pyramid is 3, and height 30; required its solidity? *Ans.* 38·97117.

3. The spire of a church is an octagonal pyramid, each side at the base being 5 feet 10 inches, and its perpendicular height 45 feet; also each side of the cavity, or hollow part, at the base is 4 feet 11 inches, and its perpendicular height 41 feet; it is required to know how many solid yards of stone the spire contains?

Ans. Solidity of the whole stone work is 32·19738 yards.

4. The height of a hexagonal pyramid is 45 feet, each side of the hexagon at the base being 10; required its solidity? *Ans.* 3897·1143.

PROBLEM VII.

To find the solidity of a cone.

RULE. Multiply the area of the base by one-third of the height, and the product will be the solidity.†

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* It is proved in every work on solid Geometry, that every pyramid is one-third of a prism having the same base and height; but the solidity of a prism is found by multiplying the area of the base by the height; therefore the solidity of a pyramid is found by multiplying the area of the base by one-third of the height.

† In solid Geometry it is proved that every cone is the third part of a cylinder having the same base and altitude; but the solidity of a cylinder is found by multiplying the area of its base by its altitude; therefore the solidity of a cone is found by multiplying the area of *its base by one-third* of its altitude.

1. The diameter of the base of a cone is 10 feet, and its perpendicular height 42 feet; what is its solidity?

$10^2 = 100 \times \cdot 7854 = 78 \cdot 54$; then $78 \cdot 54 \times \frac{42}{3} = 1099 \cdot 56$ feet.

2. The diameter of the base of a cone is 12 feet, and its perpendicular height 100; required its solidity?
Ans. 3769·92 feet.

3. The spire of a church of a conical form measures 37·6992 feet round its base; what is its solidity, its perpendicular height being 100 feet? *Ans.* 3770·1526.

4. How many cubic yards in an upright cone, the circumference of the base being 70 feet, and the slant height 30?
Ans. 134·09.

5. How many cubic feet in an oblique cone, the greatest slant height being 20 feet, the least 16, and the diameter of the base 8 feet? *Ans.* 264·73216 feet.

PROBLEM VIII.

To find the solidity of the frustrum of a pyramid.

RULE. Add the two ends and the mean proportional between them together; then multiply the sum by one-third of the perpendicular height, and the product will give the solidity.*

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* *Demonstration.* Let A^2 and a^2 be equal to the areas A B and S D; P and p the perpendiculars from the vertex V upon the planes of the bases A B and S D; and $h = P - p$ the height of the frustrum. Put also C = the entire solid. The content of the entire solid V A O B R is

$A^2 \times \frac{P}{3}$, and of the part V S Q D P $= a^2 \times \frac{p}{3}$, (Prob. VI.), and the difference between these two solids is the contents of the frustrum;

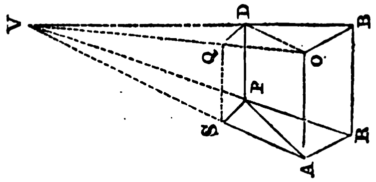
$$\text{that is, } A^2 \times \frac{P}{3} - a^2 \times \frac{p}{3} = A^2 \times \frac{h}{3} + (A^2 - a^2) \times \frac{p}{3};\text{ but}$$

$$A^2 : a^2 :: A\,R^2 : S\,P^2 \text{ (20. VI.); hence}$$

$$A : a :: A\,R : S\,P \text{ (22. VI.) But}$$

A R : S P :: R V : P V :: P : p (4. VI.); therefore by equality of ratios A : a :: P : p; and then by division $A - a : a :: h : p$; but

$$A - a : a :: A^2 - a^2 : a \times (A + a);\text{ hence}$$



1. In a square pyramid, let A O = 7, P D = 5, and the height O Q = 6; the solidity of the frustrum is required?

$7 \times 7 = 49 =$ the area of the base.

$5 \times 5 = 25 =$ the area of the section S D.

$7 \times 5 = 35 =$ the mean proportional between 49 and 25.

Therefore, $\frac{49 + 35 + 25}{3} \times 6 = 218 =$ the content of the frustrum.

2. What is the content of a pentagonal frustrum, whose height is 5 feet, each side of the base 1 foot 6 inches, and each side of the less end 6 inches?

Ans. 9·319250 cubic feet.

3. What is the content of a hexagonal frustrum, whose height is 6, and the side of the greater end 18 inches, and of the less 12 inches? *Ans.* 24·681722.

4. How many cubic feet in a squared piece of timber, the areas of the two ends being 504 and 372 inches, and its length $31\frac{1}{2}$ feet? *Ans.* 95·4 feet.

5. What is the solidity of a squared piece of timber, its length being 18 inches, each side of the greater base 18 inches, and each side of the small end 12 inches?

Ans. 28·5.

$A^2 - a^2 : A a + a^2 :: h : p$, and

$(A^2 - a^2) \times \frac{p}{3} = (A a + a^2) \times \frac{h}{3}$; therefore

$(A^2 + A a + a^2) \times \frac{h}{3} =$ the content of the frustrum, which is the *rule.*

PROBLEM IX.

To find the solidity of the frustrum of a cone.

RULE. Add the two ends and the mean proportionals between them together, then multiply one-third of the sum by the perpendicular height, and the product will be the content.*

1. How many solid feet in a tapering round piece of timber, whose length is 26 feet, and the diameters of the ends 22 and 18 inches respectively?

Here $22^2 \times \cdot 7854 = 380 \cdot 134$ inches = the area of the greater end, and

$18^2 \times \cdot 7854 = 254 \cdot 47$ inches = the area of the less end,

(Prob. XVIII. Sec. 2.) $(380 \cdot 134 \times 254 \cdot 47)^{\frac{1}{2}} = 311 \cdot 018$ = the mean proportional between the areas of the ends; then, by the rule

$$\frac{254 \cdot 47 + 380 \cdot 134 + 311 \cdot 018}{3} \times (26 \times 12) = 98345$$

cubic inches = 56·9 cubic feet, the *Ans.*

2. How many cubic feet in a round piece of timber, the diameter of the greater end being 18 inches, and that of the less 9 inches, and length 14·25? *Ans.* 14·68943 feet.

3. What is the solid content of the frustrum of a cone, whose height is 1 foot 8 inches, and the diameters of both ends 2 feet 4 inches and 1 foot 8 inches? *Ans.* 131·584.

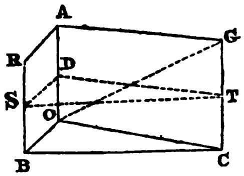
PROBLEM X.

To find the solidity of a wedge.

RULE I. Add the three parallel edges together, and multiply one-third of the sum by the area of that section of the

* The demonstration given of the last rule applies to this.

wedge which is perpendicular to these three edges, and the product will give the content.*



Note. When the quadrangular sides are parallelograms, the wedge is a triangular prism, having for its base the triangle B O C; when the quadrangles are rectangular, A O is the height of the prism, and the area of the triangle B O C multiplied by A O will give

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* *Demonstration.* Let the planes A O C G, A R B O, B R G C be trapezoids, the latter being any how inclined to the former, and C G, A O, B R parallel to one another; let also the wedge be divided into two pyramids by the plane B O G; then B is the vertex of the pyramid whose base is the triangle O G C; and G is the vertex of the pyramid which has A R B O for its base. Again, let S D be perpendicular to A O and R B; and D T perpendicular to A O and G C; then the triangular plane S D T is perpendicular to the three parallel edges B R, A O, C G. Put $p =$ the perpendicular of the triangle S D T, let fall from the vertex S upon the base D T, (produced if necessary), and P the perpendicular let fall from T, (produced if necessary); hence P and p are the heights of the pyramids A R B O G and G C O B respectively.

Now $\frac{CG \times DT}{2}$ = the area of the triangle O G C; and $\frac{GC \times TD}{2} \times \frac{p}{3} = \frac{GC}{3} \times \frac{DT \times p}{2}$ = the solid content of the pyramid O G C B, and $\frac{AO + RB}{2} \times SD$ = the area of the trapezoid A R B O; and $\frac{AO+RB}{2} \times SD \times \frac{P}{3} = \frac{AO+RB}{3} \times \frac{SD \times P}{2}$ = the solid content of the other pyramid. But the area of the triangular section S D T $= \frac{DT \times p}{2} = \frac{SD \times P}{2}$; therefore the sum $\frac{GC}{3} \times \frac{DT \times p}{2} + \frac{AO+RB}{3} \times \frac{SD \times P}{2}$, or the content of the wedge

its content; when the triangle B O C is isosceles and perpendicular to the plane A C, the wedge is of the common kind; C G is its edge, and A R B O its back.

RULE II. To twice the length of the base, add the length of the edge, multiply the sum by the breadth of the base, and the product by the height of the wedge, and $\frac{1}{6}$ of the last product will be the solidity; that is, $(2L+l) \times \frac{1}{6} b h$, by putting $L = RB$ the length of the base, $l = GC$ the length of the edge, $b = AR$ the breadth of the base, $h =$ the perpendicular height of the wedge.*

1. Let $AO = 4$, $GC = 3$, $RB = 2\frac{1}{2}$, the perpendicular $DT = 12$, and p the perpendicular distance of BR from the plane of the face $AC = 3\frac{1}{2}$ feet; required the solid content?

$$\frac{4 + 3 + 2\frac{1}{2}}{3} \times 12 \times \frac{3\frac{1}{2}}{2} = 66\tfrac{1}{2} \text{ cubic feet.}$$

2. The perpendicular height from the point T to the middle of the back A B is 24·8, the length of the edge C G 110 inches, the base R B 70 inches, and its breadth A R 30 inches; required the solidity? *Ans.* 31000 cubic inches.

3. How many cubic inches in a wedge whose altitude is 14 inches, its edge 21 inches, the length of its base 32 inches, and its breadth $4\frac{1}{2}$ inches? *Ans.* 892·5 cubic inches.

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is $\frac{GC}{3} + \frac{AO + RB}{3}$ multiplied by $\frac{DT \times p}{2}$ or by $\frac{SD \times P}{2}$; that is, one-third of the sum of the parallel edges multiplied by the area of the triangle S D T, which is the rule.

* Doctor Hutton gives the following demonstration of this rule:

Then, since it is evident that, according as the edge is shorter or longer than the base, the wedge is greater or less than half a prism of the same height and breadth with the wedge, and length equal to that of the edge, by a pyramid of the same height and breadth at the base also, and the length of whose base is equal to the difference of the length of the edge and base of the wedge; we shall have the content $\frac{1}{2} b l h + \frac{1}{3} b h \times (\pm L \mp l) = \frac{1}{2} b l h + \frac{1}{3} b h \times (L - l) = \frac{1}{6} b h \times (3l + 2L - 2l) = \overline{b h} \times (2L + l)$.

Cor. If $l = L$, the rule will become $\frac{1}{6} b h \times 3L = \frac{1}{2} b h L = \frac{1}{2}$ a prism of the same base and height, as it ought.

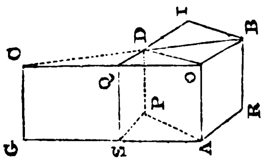
Scholium. It is evident, whether the two ends, or the two sides of the wedge be equally or unequally inclined to the base, it will make no difference in the rule.

PROBLEM XI.

To find the solidity of a prismoid, which is the frustrum of a wedge.

RULE. By either of the foregoing rules, find the solidity of two wedges whose bases are the two ends of the frustrum, and height the distance between them, and the sum of both will be the solidity of the prismoid or frustrum.*

1. In the prismoid A B P Q, there is given R B = 18, A O = 27, P D = 21, S Q = 24, B O = 12, D Q = 4, and B I = 30; what is its solidity?



$\frac{18+27+21}{3} \times \frac{30 \times 12}{2} = 3960 =$ the content of the

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* *Demonstration.* Conceive the frustrum to be cut by a plane passing through the opposite edges A O and P D, which will evidently divide it into two wedges A R B O D P and P S Q D O A. Let B I represent the perpendicular distance between the ends A R B O and S P D Q; also let B O represent the distance of A O from R B, and D Q the perpendicular distance of P D from S Q; then it is obvious that $\frac{BI \times BO}{2}$, and $\frac{BI \times DQ}{2}$ will be the respective areas of the two triangular sections of the two wedges which are perpendicular to the edges B R, A O, S Q, and P D. Then by the last rule, the content of the wedge A R B O D P will be $\frac{RB + AO + PD}{3} \times \frac{BI \times BO}{2}$; and the content of the wedge P S Q D O A will be $\frac{SQ+AO+PD}{3} \times \frac{BI \times DQ}{2}$ and the sum of both will be the *solidity of the prismoid.*

greater wedge, and $\frac{24+27+21}{3} \times \frac{30 \times 4}{2} = 1440$ the cntent of the other; then $3960 + 1440 = 5400$ the content of the frustrum.

2. What is the solidity of a piece of wood in the form of a prismoid, whose ends are rectangles, the length and breadth of one being 1 foot 2 inches and 1 foot, and the corresponding sides of the other 6 and 4 inches; the perpendicular height being $30\frac{1}{2}$ feet? *Ans.* 18·074 cubic feet.

Note. The following rule will answer for any prismoid, or cylinder, of whatever figure each end may be.

RULE. If the bases be dissimilar rectangles, take two corresponding dimensions, and multiply each by the sum of double the other dimension of the same end, and the dimension of the other end corresponding to this last dimension; then multiply the sum of the products by the height, and $\frac{1}{6}$ of the last product will be the solidity.*

PROBLEM XII.

To find the solidity of a cylindroid; or the frustrum of an elliptical cone.

RULE I. To the longer diameter of the greater end, add half the longer diameter of the less end, and multiply the sum by the shorter diameter of the greater end.

II. To the longer diameter of the less end, add half the longer diameter of the greater end, and multiply the sum by the shorter diameter of the less end.

III. Add the two preceding products together, and multiply

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* *Demonstration.* It has been shown before that the prismoid is composed of two wedges, whose bases are the two ends of the prismoid, and whose heights are equal to that of the prismoid; therefore, by the last Problem, Rule II., its solidity is $= [(2\,L + l)\,B + (2\,l + L)\,b] \times \frac{1}{6}h$; and as $\frac{1}{2}L + \frac{1}{2}l = M$, and $\frac{1}{2}B + \frac{1}{2}b = m$, are length and breadth of a section parallel to, and equally distant from, each end, we shall get

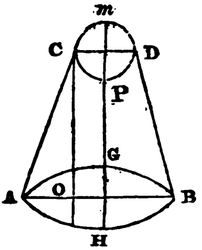
$$[(2\,L + l)\,B + (2\,l + L)\,b \times \tfrac{1}{6}h], \text{ or}$$
$$(2\,B\,L + B\,l + 2\,b\,l + b\,L) \times \tfrac{1}{6}h = (B\,L + b\,l + 4\,M\,m) \times \tfrac{1}{6}h;$$

that is, the sum of the areas of the two ends, and 4 times the section in the middle, multiplied by $\frac{1}{6}h$.

As every prismoid aud cylindroid may be conceived to consist of

the sum by ·2618 (one-third of ·7854) and then by the height, the last product will be the solidity.*

1. Let A B C D be a cylindroid, the base of which is an ellipsis, whose two diameters are 40, and 20 inches, the top a circle, whose diameter is 30 inches; what is its solidity, allowing the height to be 10 feet?



$$(AB + \tfrac{1}{2}CD) \times GH = (40 + 15) \times 20 = 1100$$
$$(CD + \tfrac{1}{2}AB) \times mP = (30 + 20) \times 30 = 1500$$
$$\text{sum} = 2600$$

Then (2600 × ·2618 × 10 = 6806·8, which divided by 144 gives 47·27 feet, the answer.

2. The transverse diameter of the greater base of a cylindroid is 13, and conjugate 8; the transverse diameter of the less base 10, and conjugate 5·2; what is the solidity of the cylindroid, its height being 12? *Ans.* 7219·3968.

3. The transverse diameter at the top of a cylindroid is 12 inches, and conjugate 7; the longer diameter at the bottom is 14 inches, and shorter 12, and its height 10 feet; required its solidity? *Ans.* 6·78 feet.

---

an infinite number of rectangular prismoids, it is evident that the last rule will answer for any prismoid or cylindroid of whatever figure the opposite end may be.

* This rule may be easily deducted from the preceding ones. Other rules are given, which find the true solidity only when the middle section between the two ends, is similar to the two ends; which never can be, except when the parallel ends are similar ellipses; that is, the transverse and conjugate diameters at each end parallel to each other; this can never happen, but when the solid is the frustrum of *an elliptical cone.*

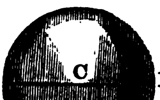
PROBLEM XIII.

To find the solidity of a sphere.

RULE I. Multiply the cube of the diameter by ·5236, and the product will be the content.

II. Multiply the diameter by the circumference of the sphere, and the product multiplied by one-sixth part of the diameter will be the solidity.*

1. Suppose the earth to be a perfect sphere, and its diameter $7957\frac{3}{4}$ miles, how many solid miles does it contain?



$7957\frac{3}{4} \times 3{\cdot}1416 =$ the circumference of the earth, (Prob. XVI. Sec. II.); then

$7957\frac{3}{4} \times 3{\cdot}1416 \times 7957\frac{3}{4} = 198943750 =$ the surface of the sphere; then

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* *Demonstration.* The reason of the first rule is evident from Solid Geometry, where it is demonstrated that a sphere is $\frac{2}{3}$ of its circumscribing cylinder. But the diameter of the base of the cylinder and its altitude are each equal to the diameter of the sphere; therefore (d being the diameter,) $d^2 \times {\cdot}7854$ is the area of the base, which being multiplied by the height of the cylinder, will give its solidity; that is, $d^2 \times {\cdot}7854 \times d = d^3 \times {\cdot}7854 =$ the content of the cylinder, the two-thirds of which will be the solidity of the sphere; that is, $d^3 \times {\cdot}7854 \times \frac{2}{3} = d^3 \times {\cdot}5236 =$ the content of the sphere, which is the first rule.

The reason of the second rule is equally obvious. For the surface of a sphere is equal to the circumference of one of its great circles multiplied by its diameter; that is, $c \times d$, c being the circumference, and d the diameter of the sphere. Now, the sphere may be considered as made up of an infinite number of pyramids, whose bases compose the surface of the sphere, and all the vertices meeting in the centre, their common altitude or height being equal to the radius of the sphere, or half the diameter; and therefore its solid content, or the solid content of any spherical pyramid, being a part contained within right lines drawn from the surface to the centre, is equal to a pyramid whose base is equal to the spherical surface, and height equal to the radius, or half the diameter; that is, $c \times d \times \frac{d}{6}$, which is the rule.—See *Solid Geometry.*

$198943750 \times 7957\frac{3}{4} \times \frac{1}{6} = 263857437760$ miles, the solidity by Rule II.

Again, $\cdot5236 \times d^3 = \cdot5236 \times (7957\frac{3}{4})^3 = 263858149120$ miles, the solidity by Rule I., which gives the result too great on account of taking ·5936 a little too great.

2. What is the solidity of a sphere, whose diameter is 24 inches? *Ans.* 7238·2464 cubic inches.

3. What is the solid content of the earth, allowing its circumference to be 25000 miles?

Ans. 263858149120 miles.

4. Required the solidity of a globe whose diameter is 30 feet? *Ans.* 14137·2.

PROBLEM XIV.

To find the solidity of the segment of a sphere.

Rule I. From three times the diameter of the sphere, deduct twice the height of the segment; multiply the remainder by the square of the height, and that product by ·5236, the last product will be the solidity.*

II. To three times the square of the radius of the segment's base, add the square of its height; multiply this sum

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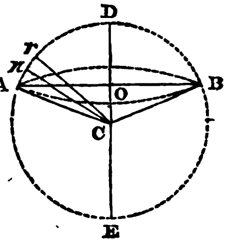
* *Demonstration.* Let A C n be a triangular pyramid, whose base n A is infinitely small; the sphere, as was said before, may be conceived to be composed of an infinite number of such pyramids, whose bases constitute the surface of the sphere, their altitudes being the radius of the sphere, and the centre their common vertex; then by the last rule, the solidity of the sphere, or any sector thereof, is equal to a pyramid, the area of whose base is the spherical surface, and its altitude the radius of the sphere.

Let O D (h) be the height of the segment A D B; then $d \times 3\cdot1416 \times h =$ the spherical surface, and $d \times 3\cdot1416 \times h \times \frac{1}{3}$ C D $= 3\cdot1416 \times dh \times \frac{1}{6} d = \cdot5236\, d^2 h =$ the solidity of the sector A C B D. But E O $\times$ O D $=$ A O^2; that is, $(d-h) \times h =$ A O^2, and A O$^2 \times 3\cdot1416 \times \frac{1}{3}$ O C $=$ the solidity of the cone A C B $= (d-h)h \times 3\cdot1416 \times \frac{1}{3}$ $(\frac{d}{2} - h) = (d-h) \times h \times \cdot5236 \times (d - 2h) = (d^2 h - 3dh^2 + 2h^3)$ $\times \cdot5236$, which taken from the solidity of the sector, leaves the solidity of the segment; that is, $(\cdot5236\, d^2 h) - (d^2 h - 3dh^2 + 2h^3) \times \cdot5236 = (3dh - 2h^3) \times \cdot5236 = (3dh - 2h^3) \times \cdot5236 = (3d$

by the height, and the product by ·5236, the last result will be the solidity.

1. What is the solidity of each of the frigid zones, the diameter of the earth being $7957\frac{3}{4}$ miles, and half the breadth, or arc of the meridian intercepted between the polar circle and the pole $23\frac{1}{2}$ degrees; that is, $AD = 23\frac{1}{2}$ degrees, supposing AB to represent the polar circle.



By Rule I.

As 1 (= tabular radius) : $3978\frac{7}{8}$ (= radius of the earth) :: ·0829399 (= tabular versed sine of $23\frac{1}{2}$ degrees) : 330·0074946, the versed sine, or height of the segment.

Then $·5236\, h^2 \times (3d - 2h) = ·5236 \times 330·0074946^2 \times 23213·2350108 = 1323679710$ the solid content.

By Rule II.

As 1 : $3978\frac{7}{8}$ :: 3987491 (= the tabular sine of $23\frac{1}{2}$ degrees) : 1586·57282526, the radius of the base.

Then $·5236\, h \times (3r^2 + h^2) = ·5236 \times 330·0074946 \times 7660544·936 = 1323680299·69$, the solidity.

2. Let A B D O be the segment of the sphere whose solidity is required. The diameter A B of the base is 16 inches, and the height O D 4 inches?

Ans. 435·6352 cubic inches.

$- 2h) \times h^2 \times ·5236$, which is the first rule. This rule will hold true, when h is less than d.

Let $r = AO$, the radius of the segment's base; then $(d - h) \times h = r^2$; hence $d = \frac{r^2 + h^2}{h}$; then substitute for d, and the rule becomes $(\frac{3r^2 + 3h^2}{h} - 2h) \times h^2 \times ·5236 = (3r^2 + h^2) \times h \times ·5236$, which is the second rule, and is always to be used when the radius of the sphere is not given. The reason of these rules might have been deduced immediately from the demonstration of the rules for performing the last Problem.

3. Required the solidity of the segment of a sphere, whose diameter is 20 feet, and the height of the segment 5 feet? *Ans.* 654·5 feet.

4. What is the solidity of a spherical segment, whose base is 16 inches and height 4? *Ans.* 435·6352.

PROBLEM XV.

To find the solidity of the frustrum or zone of a sphere.

Rule I. To the sum of the squares of the radii of the two ends, add $\frac{1}{3}$ of the square of their distance, or the height of the zone; this sum multiplied by the height of the zone, and the product again by 1·5708, will be the solidity.

II. For the middle zone of a sphere. To the square of the diameter of the end, add $\frac{2}{3}$ of the square of the height; multiply this sum by the height, and then by ·7854, the last result will be the solidity.

Or, From the square of the diameter of the sphere, deduct $\frac{1}{3}$ of the square of the height of the middle zone; multiply the remainder by the height, and then by ·7854, the last result will be the solidity.*

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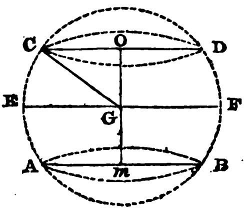
* *Demonstration.* Put H = height of the greater segment; and h = height of the less; R = radius of the greater base, and r = radius of the less. Then it is obvious that the difference between these two segments will be the zone required; that is, $(3\,R^2 + H^2) \times H \times \cdot 5236 - (3\,r^2 + h^2) \times h^2 \times \cdot 5236 = [(3\,R^2\,H + H^3 - 3\,r^2\,h + h^3)] \times \cdot 5236$. Put d = the diameter of the sphere, and then from the property of the circle, we get $(d - H) \times H = R^2$, and $(d - h) \times h = r^2$.

Hence $d = \frac{R^2 + H^2}{H}$, and $d = \frac{r^2 + h^2}{h}$, consequently $\frac{R^2 + H^2}{H} = \frac{r^2 + h^2}{h}$, and putting $a = H - h$, we get $[(3\,R^2\,H + H^3) - (3\,r^2\,h + h^3)] \times \cdot 5236 = (R^2 + r^2 + \frac{1}{3}\,a^2) \times a \times 1 \cdot 5708$. Now, if one of the radii pass through the centre, we get $R^2 = \frac{d^2}{4} = C\,O^2 + G\,O^2 = r^2 + a^2$; hence the last theorem becomes $(r^2 + \frac{2}{3}\,a^2) \times a \times 3 \cdot 1416 = (\frac{1}{4}\,d^2 - \frac{1}{3}\,a^2) \times a \times 3 \cdot 1416$. Hence $(r^2 + \frac{2}{3}a^2) \times a \times 6 \cdot 2832 = (\frac{1}{4}\,d^2 = \frac{1}{3}\,a^2) \times a \times 6 \cdot 2832$ will express the solidity of the middle zone



1. Required the solidity of the frustrum of a sphere, the diameter of whose greater end is 4 feet, the diameter of the less end 3 feet, and the height $2\frac{1}{2}$ feet?

$(2^2 + 1\cdot5^2 + \frac{1}{3} \times 2\cdot5^2) \times 1\cdot5708 \times 2\cdot5 = 8\frac{1}{3} \times 3\cdot927 = 32\cdot725$, the solidity of the frustrum.

2. What is the solidity of the temperate zone, its breadth being 43 degrees, the radius of the top being 1586·57282526, and the radius of the base 3648·86750538, and height 2062·2655?

$(3648\cdot86750538^2 + 1586\cdot57282526^2 + \frac{1}{3} \times 2062\cdot2655^2) \times 2062\cdot2655 \times 1\cdot5708 = 17249136 \times 2062\cdot2955 \times 1\cdot5708 = 55877778668$, the solidity of each temperate zone.

3. Required the solidity of the torrid zone, which extends $23\frac{1}{2}$ degrees on each side of the equator, the diameter being $7957\frac{3}{4}$ miles, and height 3173·14565052?

$(7957\cdot75^2 - \frac{1}{3} \times 3173\cdot14565052^2) \times 3173\cdot14565052 \times \cdot7854 = 149455081137$, the answer.

4. What is the solidity of the middle zone of a sphere, whose top and bottom diameters are each 3 inches, and height 4 inches? *Ans.* 61·7848.

5. What is the solid content of a zone, whose greater diameter is 20 feet, less diameter 15 feet, and the height 10 feet? *Ans.* 189·58.

6. How many solid feet in a zone, whose greater diameter is 12 feet, and less diameter 10; the height being 2? *Ans.* 195·8264.

A B D C, being double the former, where a is $\frac{1}{2}$ the altitude, and $r =$ half the diameter of each end. Put $A =$ the whole altitude, and $d = 2r$, the diameter of each end; and the theorem becomes

$(d^2 + \frac{2}{3} A^2) \times A \times \cdot7854 = (d^2 - \frac{2}{3} A) \times A \times \cdot7854.$

The reason of this rule may be deduced from the demonstration given for finding the solidity of the whole sphere.

PROBLEM XVI.

To find the solidity of a circular spindle.

RULE. Find the distance of the chord of the generating circular segment from the centre of the circle, and also the area of this segment.

Then, from $\frac{1}{3}$ of the cube of half the length of the spindle, or half chord of the segment, subtract the product of the central distance, and half the area of the segment; the remainder, multiplied by 12·5664 will give the solidity.*

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* *Demonstration.* Put $FC = a$, $FS = c$, and $r =$ radius; conceive an infinite number of ordinates y, y, y, &c., to be drawn, as in the figure, and let the distance between every two of them be represented by x. Then from the property of the circle, $TO \times OB = OV^2 = a^2$; that is, $(r + c + y') \times OB = x^2$, that is, $(c + OB + y' + c + y') \times OB = (2c + OB + 2y') \times OB = x^2$. But $(c + OB + y')^2 = r^2$; therefore, $(c + OB + y')^2 - (2c + OB + 2y') \times OB = r^2 - x^2 = c^2 + 2cy' + y'^2$; hence, $r^2 - c^2 - 2cy' - x^2 = y'^2$; but $SC^2 (r^2) - FS^2 (c^2) = FC^2 (a^2)$; therefore, $a^2 - 2cy' - x^2 = y'^2$. Now, if we take $x = 0, 1, 2, 3$, &c., we get

$$a^2 - 2cy = y^2$$

$$a^2 - 2cy' - 1^2 = y'^2$$

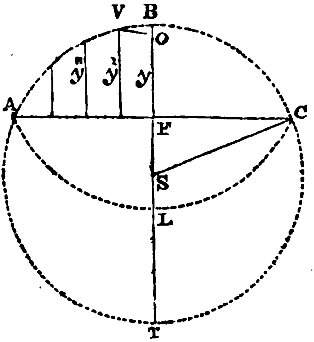
$$a^2 - 2cy'' - 2^2 = y''^2$$

$$a^2 - 2cy''' - 3^2 = y'''^2$$

&c. &c. &c. &c.

$$a^2 - 2c \times 0 - a^2 = 0^2$$

But if we conceive the spindle to revolve about the chord A C, the sum of all the circles whose radii are y, y', y'', &c., will be the solidity of A B L, and the area of these circles is, $\{(2y)^2 [= 4y^2] + (2y')^2$



1. Let the axis A C of a circular spindle be 40 inches, and its greater diameter B L 30 inches; what is its solidity?

$20^2 \div 15 = 26\frac{2}{3}$, then $26\frac{2}{3} + 15 = 41\frac{2}{3}$, the diameter of the circle. Again $\frac{41\frac{2}{3}}{2} - \frac{30}{2} = 5\frac{5}{6}$, the central distance.

Now $15 \div 41\frac{2}{3} = \cdot 36$, the area segment corresponding to which is ·254550, which multiplied by the square of $41\frac{2}{3}$ produces 441·92708 the area of the generating segment A B C, the half of which is 220·96354.

Lastly, $(20^2 \div 3) - (5\frac{5}{6} \times 220 \cdot 96354) = 1377 \cdot 71268$, and this multiplied by 12·5664 produces 17312·88862 cubic inches, the solidity required.

$[= 4y'^2] + \&c.\} \times \cdot 7854 = (y^2 + y'^2 + y''^2 + \&c.) \times 4 \times \cdot 7854 =$ $(y^2 + y'^2 + y''^2 + \&c.)\ 3 \cdot 1416$. But $y^2 + y'^2 + y''^2 + \&c.$ = the sum of the left-hand members of the equations; therefore the sum of the left-hand members multiplied by 3·1416 will give the solidity of the part A B L. The sum $a^2 + a^2 + a^2$ &c. $= a^3$, and of $2cy + 2cy' + 2cy'' + \&c. = 2c \times$ A B F; also, $0^2 + 1^2 + 2^2 + 3^2 + \ldots a^2 = \frac{a^3}{3}$; therefore the sum of the left-hand members of the equations is $a^3 - 2c \times$ space A B F $- \frac{a^3}{3} = \frac{2a^3}{3} - 2c \times$ space A B F $= \left(\frac{a^3}{3} - c \times \text{space A B F}\right) \times 2$. Then, $\left(\frac{a^3}{3} - c \times \text{space A B F}\right) \times 2 \times 3 \cdot 1416$ = the solidity of A B L, which is half the spindle, therefore the whole spindle will be $\left(\frac{a^3}{3} - c \times \text{space A B F}\right) 4 \times 3 \cdot 1416 = \left(\frac{a^3}{3} - c \times \text{space A B F}\right) \times 12 \cdot 5664$, which is the rule.

2. The longest diameter of a circular spindle is 48, and the middle diameter 36; required the solidity of the spindle?

Ans. 29866·6634.

PROBLEM XVII.

To find the solidity of the middle frustrum of a circular spindle.

RULE I. Find the distance of the centre of the middle frustrum, from the centre of the circle.

II. Find the area of a segment of a circle, the chord of which is equal to the length of the frustrum, and height half the difference between its greatest and least diameters; to which add the rectangle of the length of the frustrum and half its least diameter; the result will be the generating surface.

III. From the square of the radius, subtract the square of the central distance, the square root of the remainder will give half the length of the spindle.

IV. From the square of half the length of the spindle take $\frac{1}{3}$ of the square of half the length of the middle frustrum, and multiply the remainder by the said half length.

V. Multiply the central distance by the generating surface, and subtract this product from the preceding; the remainder, multiplied by 6·2832, will give the solidity.*

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* *Demonstration.* By the last, we have $r^2 - x^2 = (c + y')^2 = c^2 + 2cy' + y'^2$, and $r^2 - c^2 - 2cy' - x^2 = y'^2$; put $r^2 - c^2 = a^2$; then

$$a^2 - 2cy' - x^2 = y'^2$$

Put 0, 1, 2, 3, &c., for x, then

$$a^2 - 2cy - 0^2 = y^2$$

$$a^2 - 2cy' - 1^2 = y'^2$$

$$a^2 - 2cy'' - 2^2 = y''^2$$

$$a^2 - 2cy''' - 3^2 = y'''^2$$

&c. &c. &c. &c.

$$a^2 - 2cy - Dx^2 = y^2$$

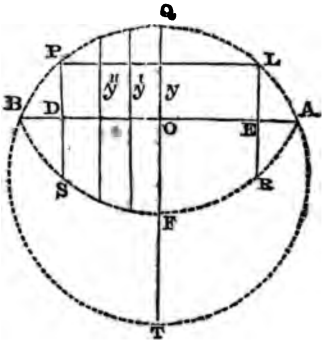
1. Required the solidity of the middle frustrum of a circular spindle, the length D E being 40, the greatest diameter Q F 32, and the least diameter P S 24.

First, $20^2 \div 4 = 100$, and $100 + 4 = 104$, the diameter of the circle.

Again, $52 - 16 = 36$, the central distance. Also, $\frac{1}{2}(32 - 24) = 4$, and $4 \div 104 = \cdot 038\frac{6}{13}$, the area segment corresponding to which is ·009940, which multiplied by the square of 104, produces 107·51104, the area of P L Q; and $40 \times 12 = 480$ the area of the rectangle P D E L.

Hence $107{\cdot}51104 + 480 = 587{\cdot}51104$ the area of the generating surface P D L E.

Next $\sqrt{(52^2 - 36^2)} = \sqrt{(1408)} = 8\sqrt{(22)} =$ B O half the length of the spindle;



From what has been said in the last demonstration, we have $a^2 \times Dx - 2c \times \text{space DPQO} - \frac{Dx^3}{3} =$ sum of the left-hand members of the equation. But

$$a^2 \times Dx - 2c \times \text{space DPQO} - \frac{Dx^3}{3} =$$
$$(a^2 - Dx^2) \times Dx - 2c \times \text{space DPQO} =$$
$$(a^2 - Dx^2) \times Dx - c \times \text{space PDEL}.$$

Then this multiplied by $2 \times 3{\cdot}1416$ will, from what has been said in the last, give the solidity of the frustrum; that is, $(a^2 - Dx^2) \times Dx - c \times \text{space PDEL} \times 6{\cdot}2832$, which is the rule.

And $(1408 - \frac{400}{3}) \times 20 = 25493\frac{1}{3}$.

Then $36 \times 587{\cdot}51104 = 21150{\cdot}39744$, and $(25493\frac{1}{3} - 21150{\cdot}39744) \times 6{\cdot}2832 = 27287{\cdot}5347$, the required solidity.

2. What is the solidity of the middle frustrum P S R L of a circular spindle, whose middle diameter F Q is 36, the diameter P S of the end 16, and its length D E 40?

Ans. 29257·2904.

3. If a cask in the form of the middle frustrum of a circular spindle, have its head diameter 24, bung diameter 32, and length 40 inches; how many ale gallons does it hold?

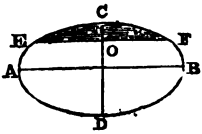
Ans. Its solid content in inches is 27286·5411256, and the content in gallons is 96·7608, allowing 282 cubic inches to the ale gallon.

PROBLEM XVIII.

To find the solidity of a spheroid.

RULE. Multiply the square of the revolving axis by the fixed axis, and this product again by ·5236 for the solidity.*

1. What is the solidity of a prolate spheroid whose longer axis A B is 55 inches, and shorter axis C D 33?



Here $33^2 \times 55 \times {\cdot}5236 = 31361{\cdot}022$ cubic inches, the answer.

2. What is the solidity of an oblate spheroid, whose longer axis is 100 feet, and shorter axis 6?

Ans. 31416 cubic feet.

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* *Demonstration.* It was shown in Proposition XIII. Cor. 2, Ellipse, that the solidity of the spheroid is $\frac{2c^2 tn}{3}$, when t is the

3. What is the solidity of a prolate spheroid, whose axes are 40 and 50? *Ans.* 41888.

4. What is the solidity of an oblate spheroid, whose axes are 20 and 10? *Ans.* 2094·4.

PROBLEM XIX.

To find the solidity of the segment of a spheroid, the base of the segment being parallel to the revolving axis of the spheroid.

CASE I.

RULE. From three times the fixed axis, deduct twice the height of the segment, multiply the remainder by the square of the height, and that product by ·5236.

Then say, as the square of the fixed axis, is to the square of the revolving axis; as the product, found above, is to the solidity of the spheroidal segment.*

1. What is the content of the segment of a prolate spheroid, the height O C being 5, the fixed axis 50, and the revolving axis 30.—See last figure.

$50 \times 3 - 5 \times 2 = 150 - 10 = 140$; then
$140 \times 5^2 = 3500$, and $3500 \times \cdot5236 = 1832\cdot6$; then
$25 : 9 :: 1832\cdot6 : 659\cdot736$ the answer.

---

transverse, c the conjugate, and $n = \cdot7854$; but $\frac{2c^2tn}{3} = \frac{2}{3}n \times c^2 t = \frac{2}{3} \times \cdot7854 \times c^2 t = \cdot5236 \times c^2 t$, which is the rule.

The solidity of the oblate spheroid is $\frac{2nt^2c}{3} = \frac{2}{3}n \times t^2 c = \cdot5236 \times t^2 c$. See Prop. XIV. Cor. 1, Ellipse. See also Prop. XIII. Cor. II.

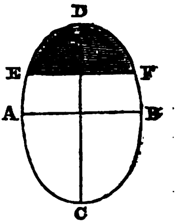
* *Demonstration.* It is proved in Proposition XIII. Cor. 4, Ellipse, that the solidity of the segment is $\frac{2nc^2}{3t^2} \times (3tx^2 - 2x^3)$, when x is the abscissa or height of the segment, y the corresponding ordinate, t the transverse, and c the conjugate; but
$\frac{2nc^2}{3t^2} \times [(3t - 2x) \times x^2] = \frac{c^2}{t} \times \frac{2}{3}n \times [(3t - 2x) \times x^2] = \frac{c^2}{t} \times$
$\cdot5236 \times [(3t - 2x) \times x^2]$, which is the rule.

CASE II.

When the base is elliptical, or perpendicular to the revolving axis.

RULE. From three times the revolving axis, take double the height; multiply that difference by the square of the height, and the product again by ·5236.

Then as the revolving axis is to the fixed axis, so is the last product to the content.



2. What is the content of the segment of a spheroid, whose fixed axis is 50, revolving axis 30, and height 6?

$30 \times 3 - 2 \times 6 = 90 - 12 = 78$;

Then $78 \times 6^2 = 2808$; and $2808 \times \cdot 5236 = 1470 \cdot 2688$;

Then $30 : 50 :: 1470 \cdot 2688 : 2450 \cdot 448$, the answer.

3. In a prolate spheroid, the transverse or fixed axis is 100, the conjugate or revolving axis is 60, and the height of the segment 10; required the solidity? *Ans.* 5277·888.

4. If the axes of a prolate spheroid be 10 and 6, required the content of the segment, whose height is 1, and its base parallel to the revolving axis? *Ans.* 5·277888.

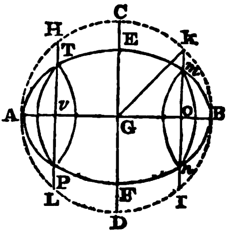
PROBLEM XX.

To find the solidity of the middle zone of a spheroid, the diameter of the ends being perpendicular to the fixed axis, the middle diameter, and that of either end being given, together with the length of the zone.

RULE. To twice the square of the middle diameter, add the square of the diameter of the end; multiply the sum by the length of the zone, and the product again by ·2618 for the solidity.*

---

* *Demonstration.* The solidity of the middle zone of a sphere (by Problem XV. Section 4.) is $(d'^2 + \frac{2}{3} A^2) \times A \times \cdot 7854$.



1. What is the solidity of the middle zone of an oblate spheroid, the middle diameter being 100, the diameter of the end 80, and the length 36?

$100^2 \times 2 + 80^2 = 26400$; then $26400 \times 36 = 950400$, and $950400 \times \cdot 2618 = 248814\cdot 72$, the answer.

2. What is the solidity of the middle frustrum of a spheroid, the greater diameter being 30, the diameter of the end 18, and the length 40?

Ans. 22242·528.

Now let $f =$ A B the fixed axis, } For the oblong spheroid.
$r =$ E F the revolving axis, }
$h =$ V O the height of the middle frustrum,
$D =$ K I the diameter of one end of the spherical zone,
$d = m\,n$ the corresponding diameter of the spheroidal zone.

Then by substitution the above equation for the spherical zone will become $\left(D^2 + \frac{2h^2}{3}\right) \times h \times \cdot 7854 = (3D^2 + 2h^2) \times h \times \cdot 2618$.

But $KO^2 = GK^2 - GO^2$; that is, $\frac{D^2}{4} = \frac{f^2}{4} - \frac{h^2}{4}$. Hence $D^2 = f^2 - h^2$; and consequently the solidity of the spherical zone is $(3f^2 - h^2) \times h \times \cdot 2618$. But it has been shown that $\cdot 5236 f^3 : \cdot 5236 f r^2 :: (3f^2 - h) \times h \times \cdot 2618$: the solidity of the spheroidal frustrum; that is, $f^2 : r^2 :: (3f^2 - h^2) \times h \times \cdot 2618$: the solidity of the spheroidal frustrum.

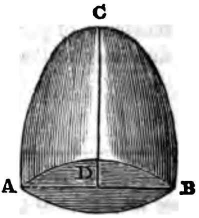
By Proposition XIV. Cor. 2, Ellipsis, $CD^2 : EF^2 :: IK^2 : mn^2$; that is, $f^2 : r^2 :: f^2 - e^2 : d^2$; hence $f^2 = \frac{r^2 h^2}{r^2 - d^2}$; therefore $\frac{r^2 h^2}{r^2 - h^2}$: $r^2 :: \left(\frac{3r^2 h^2}{r^2 - d^2} - h^2\right) \times h \times \cdot 2618$: the solidity of the spheroidal frustrum $= (2r^2 + d^2)\,h \times \cdot 2618$.

Putting $r =$ AB, the rule for the spheroidal zone will be the same.

PROBLEM XXI.

To find the solidity of a parabolic conoid.

Rule. Multiply the square of the diameter of its base by ·3927, and that product by the height; the last product will be the solidity.*



1. What is the solidity of the parabolic conoid, whose height is 10 feet, and the diameter of its base 10 feet?

$10^2 \times \cdot 3927 = 39\cdot 27$; then $39\cdot 27 \times 10 = 392\cdot 7$ the solidity required.

2. What is the solidity of a parabolic conoid, whose height is 30, and the diameter of its base 40? *Ans.* 18849·6.

3. What is the content of the parabolic conoid whose altitude is 40, and the diameter of its base 12? *Ans.* 2261·952.

4. Required the solidity of a parabolic conoid, whose height is 30, and the diameter of its base 8? *Ans.* 753·984.

PROBLEM XXII.

To find the solidity of the frustrum of a parabolic conoid.

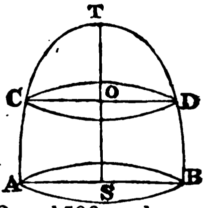
Rule. Multiply the sum of the squares of the diameters of the two ends by the height, and that product by ·3929; the last product will be the solidity.†

---

* *Demonstration.* It was proved in Prop. VIII. *Arith. of Infinities*, that the parabolic conoid is half of a cylinder of the same base and height. But the solidity of the cylinder is $D^2 \times \cdot 7854 \times h$, ($h$ being the height of the cylinder); therefore the solidity of the conoid is

$$\frac{D^2 \times \cdot 7854 \times h}{2} = D^2 \times \cdot 3927 \times h, \text{ which is the rule.}$$

† *Demonstration.* It was proved in Prob. VIII. Cor. *Arith. of Inf.* that the solidity of the lower frustrum of a parabolic conoid is equal to half the sum of both bases multiplied by the height of the frustrum. If D be the diameter of the greater base, and d the diameter of the less, their areas are $D^2 \times \cdot 7854$, and $d^2 \times \cdot 7854$; then all



1. The greater diameter of the frustrum is 10, and the less diameter 5; what is the solidity, the length being 12?

$$10^2 = 100$$
$$5^2 = 25$$

125. Then $125 \times 12 = 1500$, and $1500 \times \cdot 3927 = 589\cdot05$ the solidity.

2. The greater diameter of the frustrum of a parabolic conoid is 20, the less 10, and its height 12; what is the solidity? *Ans.* 2356·2.

3. The greater base of the frustrum of a parabolic conoid is 30, the less 10, and the height 50, required the solidity? *Ans.* 19635.

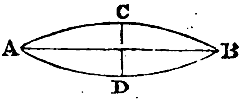
4. The greater base of the frustrum of a parabolic conoid is 15, the less base 12, and the height 8; required the solidity? *Ans.* 1159·2504.

PROBLEM XXIII.

To find the solidity of a parabolic spindle.

Rule. Multiply the square of the middle diameter by ·7854, and that product by the length; then $\frac{8}{15}$ of this product will be the solidity.*

1. The middle diameter C D, of a parabolic spindle is 10 feet, and the length A B is 40; required its solidity?



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their sum is $(D^2 + d^2) \times \frac{\cdot 7854}{2} = (D^2 + d^2) \times \cdot 3927$, which multiplied by the height of the frustrum will give its solidity, viz. $(D^2 + d^2) \times \cdot 3927 \times h$, is the solidity, which is the rule.

* *Demonstration.* It was shown in Prop. IX. *Arith. of Infinities*, that every parabolic spindle is equal to $\frac{8}{15}$ of its circumscribing cylinder. But, the contents of the circumscribing cylinder is $D^2 \times \cdot 7854 \times l$, D being the middle diameter, and l the length; therefore $\frac{8}{15} \times D^2 \times \cdot 7854 \times l$, is the solidity of the spindle, which is the rule.

$10^2 \times \cdot 7854 \times 40 = 3141 \cdot 6$ feet.

Then $\frac{8}{15} \times 3141 \cdot 6 = 1675 \cdot 52$ feet, the answer.

2. The middle diameter C D, of a parabolic spindle is 12 feet, and the length A B is 30, required the solidity? *Ans.* 1809·5616.

2. The middle diameter of a parabolic spindle is 3 feet, and the length 9 feet; required its solidity? *Ans.* 33·92928.

3. The middle diameter of a parabolic spindle is 6 feet, and the length 10; required its solidity? *Ans.* 150·7968.

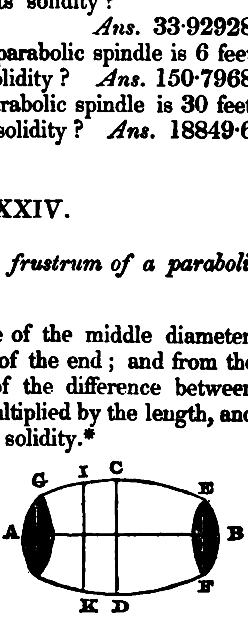
4. The middle diameter of a parabolic spindle is 30 feet, and the length is 50; required its solidity? *Ans.* 18849·6.

PROBLEM XXIV.

To find the solidity of the middle frustrum of a parabolic spindle.

RULE. To double the square of the middle diameter, add the square of the diameter of the end; and from the sum subtract $\frac{4}{10}$ of the square of the difference between these diameters; the remainder multiplied by the length, and that product by ·2618, will be the solidity.*

1. In a parabolic spindle, the middle diameter of the middle frustrum is 16, the least diameter 12, and the length 20; required the solidity of the frustrum?



Here $2 \times 16^2 + 12^2 - \frac{4}{10} \times 4^2 = 512 + 144 - 6 \cdot 4 = 649 \cdot 6$; hence $649 \cdot 6 \times 20 \times \cdot 2618 = 3401 \cdot 3056$, the solidity.

2. The bung diameter of a cask is 30 inches, the head

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* *Demonstration.* In Prop. IX. Cor. *Arith. of Infinities*, the equation for the solidity of the frustrum is $(2\,D^2 + C^2 - \frac{4}{10}\,d^2) \times L \times \cdot 2618$, where D = middle diameter C = diameter of the end, d = difference of diameters, and L = the length, which is the rule.

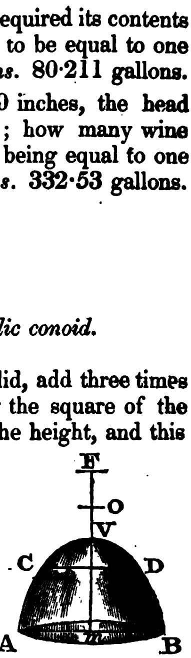
diameter 20 inches, and the length 40; required its contents in ale gallons, allowing 282 cubic inches to be equal to one gallon? *Ans.* 80·211 gallons.

3. The bung diameter of a cask is 40 inches, the head diameter 30 inches, and the length 60; how many wine gallons does it contain, 231 cubic inches being equal to one gallon? *Ans.* 332·53 gallons.

PROBLEM XXV.

To find the solidity of a hyperbolic conoid.

RULE. To double the height of the solid, add three times the transverse axis, multiply the sum by the square of the radius of the base, and that product by the height, and this last product by ·5236; the result divided by the sum of the height and transverse axis, will give the solidity.*



1. Required the solidity of an hyperbolic conoid, whose height V m is 50, the diameter A B 103·923048, and the transverse axis V E 100?

Here $(2 \times 50 + 3 \times 100) \times \frac{(103{\cdot}923048)^2}{2} = 400 \times$

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* *Demonstration.* Put A m = R, and x = V m the height; then $R^2 \times 3{\cdot}1416$ = area of the base, and $R^2 \times x \times 3{\cdot}1416$ = solidity of the cylinder of the same base and height as the hyperbolic conoid. But by Proposition VII., (Hyperbola,) the hyperbolic conoid is to the cylinder of the same base and height as $\frac{1}{2}t + \frac{1}{3}x$ to $t + x$; therefore $t + x : \frac{1}{2}t + \frac{1}{3}x :: R^2 \times x \times 3{\cdot}1416$: hyperbolic conoid, or

$t + x : \frac{3t + 2x}{6} :: R^2 \times x \times 3{\cdot}1416$: hyperbolic conoid, which is

equal $\frac{(3t + 2x)}{t + x} \times R^2 \times x \times \frac{3{\cdot}1416}{6} = \frac{(3t + 2x) \times R^2 \times x \times {\cdot}5236}{t + x}$,

which is the rule, t being the transverse axis.

$2700 = 1080000$; and $\frac{1080000 \times 50 \times \cdot 5236}{150} = 188496$, the solidity.

2. What is the contents of an hyperboloid, whose altitude is 10, the radius of its base 12, and the transverse 30?

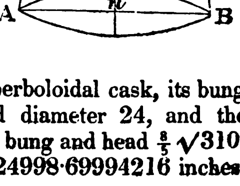
Ans. 2073·451151369.

PROBLEM XXVI.

To find the solidity of the frustrum of an hyperboloid, or hyperbolic conoid.

RULE. To four times the square of the middle diameter, add the sum of the squares of the greatest and least diameters; multiply the result by the altitude, and that product by ·1309, for the solidity.*

1. Required the solidity of the frustrum A C E H D B of an hyperbolic conoid, whose greatest diameter A B is 96, least diameter E H 54, middle diameter C D 76·4264352, and the altitude $m\,n$ 25?



Here $4\,CD^2 + AB^2 + EH^2 = (5841 \times 4) + 9216 + 2916 = 35496$, and $35496 \times 25 \times \cdot 1309 = 116260\cdot 66$, the answer.

2. What is the solidity of an hyperboloidal cask, its bung diameter being 32 inches, its head diameter 24, and the diameter in the middle between the bung and head $\frac{8}{5}\sqrt{310}$, and its length 40 inches? *Ans.* 24998·69994216 inches.

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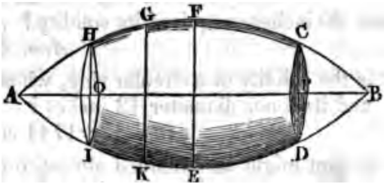
* This rule is demonstrated in the note to Problem XI. Sec. IV., where it is shown that four times the area in the middle, added to the areas of the two ends, and the sum multiplied by $\frac{1}{6}$ of the height, gives the solidity, but ·1309 being ·7854 ÷ 6, then $(4\,D^2 + d^2 + d'^2) \times h \times \frac{\cdot 7854}{6} = (4\,D^2 + d^2 + d'^2) \times h \times \cdot 1309$, which is the rule.

PROBLEM XXVII.

To find the solidity of a frustrum of an elliptical spindle, or any other solid formed by the revolution of a conic section about an axis.

RULE. Add together the square of the greatest and least diameters, and the square of double the diameter in the middle between the two; multiply the sum by the length, and the last product by ·1309 for the solidity.*

1. What is the content of the middle frustrum C D I H of any spindle, the length O P being 40, the greatest, or



middle diameter E F 32, the least, or diameter at either end C D 24, and the diameter G K 30·157568?

Here $32^2 + (2 \times 30{\cdot}157568)^2 + 24^2 = 5237{\cdot}89$ sum;
Then $5237{\cdot}89 \times 40 = 209515{\cdot}6$, and
$209515{\cdot}6 \times {\cdot}1309 = 27424$, the answer.

2. What is the content of the segment of any spindle, the length being 20, the greatest diameter 10, the least diameter at either end 5, and the diameter in the middle between these 8? *Ans.* 997·458.

PROBLEM XXVIII.

To find the solidity of a circular ring.

RULE. To the thickness of the ring add the inner diameter; multiply the sum by the square of the thickness, and the product by 2·4674, for the solidity.†

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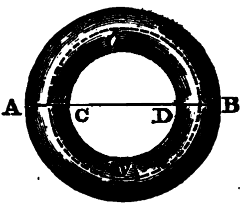
* This rule is the same as the last, and is demonstrated in Problem XI. Section IV.

† *Demonstration.* Let *m o n v* be a cylindrical ring, the diameter



1. The thickness of a cylindrical ring is 2 inches, and the diameter C D 5 inches; required its solidity?

$(2+5) \times 4 = 28$; then $28 \times 2{\cdot}4674 = 69{\cdot}0872$ cubic inches, the answer.

2. Required the solidity of an iron ring whose axis forms the circumference of a circle; the diameter of a section of the ring 2 inches, and the inner diameter, from side to side, 18 inches? *Ans.* 197·3925 cubic inches.

3. The thickness of a cylindrical ring is 7 inches, and the inner diameter 20 inches, required its solidity? *Ans.* 3264·3702.

4. What is the solidity of a circular ring, whose thickness is 2 inches, and its inner diameter 12 inches? *Ans.* 138·1744 cubic inches.

The last section might be extended almost to any length; the various ways of dividing areas, solids, and their surfaces, are endless, and may form an agreeable task for the mathematician; but as our object is to promote the useful and not the speculative parts of the science, we have omitted such problems as are remarkable only for their curiosity. We shall therefore proceed to treat of the five regular bodies, sometimes called the five Platonic bodies, from their having been invented by Plato, who conceived many curious mysteries annexed to them.

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of a section of which is A C, and the mean length *m o n v* passing through its centre. Then $AC + CD = mn$; therefore the mean length of the cylinder is $mn \times 3{\cdot}1416$; and the area of a section A C is $AC^2 \times {\cdot}7854$; but the solidity of a cylinder is found by multiplying the area of its base, which is here $AC^2 \times {\cdot}7854$, by its length which is $mn \times 3{\cdot}1416$; that is, $AC^2 \times {\cdot}7854 \times mn \times 3{\cdot}1416 = AC^2 \times (AC + CD) \times 2{\cdot}4674$, which is the rule.

THE FIVE REGULAR BODIES.

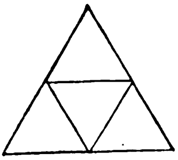
SECTION VI.

DEFINITIONS.

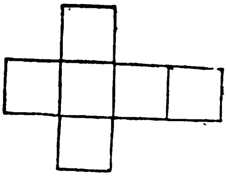
A Regular Body is a solid contained under a certain number of similar and equal plane figures.

Only five regular bodies can possibly be formed. Because it is proved in Solid Geometry that only three equi-lateral and equi-angular plane figures, joined together, can make a solid angle.

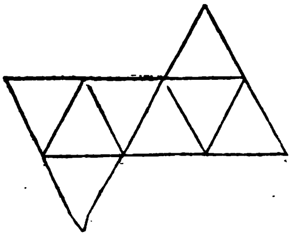
1. The *tetraedron*, or equi-lateral pyramid, is a solid having four triangular faces.



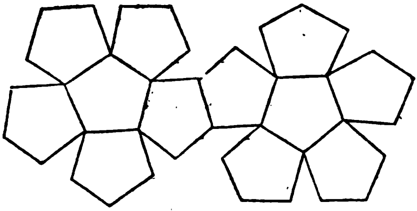
2. The *hexaedron*, or cube, is a solid having six square faces.



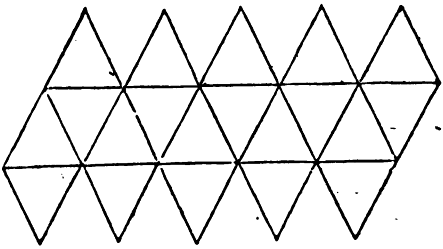
3. The *octaedron* is a regular solid having eight triangular faces.



4. The *dodecaedron* has twelve pentagonal faces.



5. The *icosaedron* has twenty triangular faces.



PROBLEM I.

To find the solidity of a tetraedron.

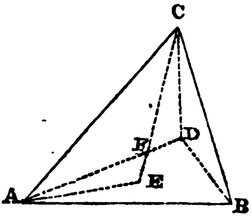
RULE I. Multiply $\frac{1}{12}$ of the cube of the lineal side by the root of 2, and the product will be the solidity.

II. Or, generally, multiply the cube of the length of a side of the body by the tabular solidity, and the product will be the solidity of the body.*

---

* *Demonstration.* Let A B C be the tetraedron; from C let fall the perpendicular C E on the opposite side A B D, and join E A. Then $AC^2 = AE^2 + EC^2$; but $\frac{1}{3} AC^2 = \frac{1}{3} AB^2 = AE^2$; therefore $\frac{2}{3} AC^2 = EC^2$. Hence, $AC \sqrt{\frac{2}{3}} = EC$; but $ABD = \frac{1}{4} AB^2 \sqrt{3} = \frac{1}{4} AC^2 \sqrt{3}$; then the solidity will be equal to the area of the base multiplied by $\frac{1}{3}$ of the altitude, (*Solid Geometry*); that is, $\frac{1}{3} CE \times ABD = \frac{1}{3} AC \sqrt{\frac{2}{3}} \times \frac{1}{4} AC^2 \sqrt{3} = \frac{1}{12} AC^3 \sqrt{2}$, the solidity, which is the rule.

The reason of the second rule is obvious from a property in Solid



1. If the side of each face of a tetraedron be 1, required its solidity?

Here $\frac{1}{12} \times 1^3 \times \sqrt{2} = \frac{1}{12} \times \sqrt{2} = \cdot 11785113$, the solidity.

2. The side of a tetraedron is 12, what is its solidity?

Ans. 203·6467.

PROBLEM II.

To find the solidity of a hexaedron, or a cube.

RULE. Cube the side for its solidity.*

1. If the linear side of an hexaedron be 3, what is its content? *Ans.* $3 \times 3 \times 3 = 27$.

PROBLEM III.

To find the solidity of an octaedron.

RULE. Multiply the cube of the side by the square root of 2, and $\frac{1}{3}$ of the product will be the content.†

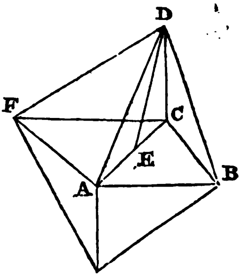
1. What is the solidity of an octaedron, when the linear side is 1?

Geometry, viz. that similar solids are to one another as the cubes of their like sides; and the tabular number being the contents of solids whose sides are 1; therefore the cube of any side multiplied by the tabular number corresponding to the figure, will give its solidity.

* This rule is demonstrated in Problem I. Section V.

† Let E be the centre of the solid, or the middle of the diagonal A C, join D E, which is equal to A E.

The solid is evidently composed of two equal square pyramids, each base of which, A B C F, is equal to the square of the linear side of the solid, the altitude of each being equal to D E, or A E, half the diagonal of that square. Now $AB^2 = ABCF$; but the area $ABCF \times \frac{2}{3} AE = AB^2 \times \frac{1}{3} AC = \frac{1}{3} AB^2 \sqrt{(AB^2 + BC^2)} = \frac{1}{3} AB^2 \sqrt{(2 AB^2)} = \frac{1}{3} AB^3 \sqrt{2}$, which is the rule.



$1^3 \times \sqrt{2} \times \frac{1}{3} = \frac{1}{3}\sqrt{2} =$ ·4714045.

2. What is the solidity of the octaedron, whose linear side is 2?

Ans. 3·7712.

PROBLEM IV.

To find the solidity of a dodecaedron.

RULE. To 21 times the root of 5 add 47, and divide the sum by 40; multiply the root of the quotient by 5 times the cube of the lineal side, and the product will be the solidity.*

---

* *Demonstration.* Let A represent a solid angle of the dodecaedron, and connect the extremities of the sides A B, A C, A D, of the faces which form the angle, by the lines B C, C D, and D B, forming an equilateral triangle B C D, within the solid, on the centre of which, let fall the perpendicular A E; join the centre F, of one of the faces, and the points A and C.

The angle C A D contains 108 degrees, the sine of which is $\frac{1}{4}\sqrt{(10+2\sqrt{5})}$ to the radius 1.

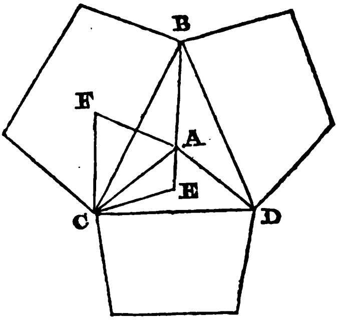
The angle A D C contains 36 degrees, the sine of which is $\frac{1}{4}\sqrt{(10-2\sqrt{5})}$. (See *Trigonometry*).

Hence $\sqrt{(10-2\sqrt{5})} : \sqrt{(10+2\sqrt{5})} :: \text{A C} : \text{D C} =$

$$\text{A C}\sqrt{\frac{10+2\sqrt{5}}{10-2\sqrt{5}}} = \text{A C}\sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = \text{A C}\sqrt{\frac{(5+\sqrt{5})^2}{25-5}} = \text{A C} \times \frac{5+\sqrt{5}}{2\sqrt{5}} = \frac{1+\sqrt{5}}{2} \times \text{A C}.$$

In somewhat a similar manner, we find $\text{C E} = \frac{1}{3}\text{C D}\sqrt{3} = \text{C D}\sqrt{\frac{1}{3}} = \frac{1+\sqrt{5}}{2\sqrt{3}} \times \text{A C}$; hence $\text{E A} = \sqrt{(\text{A C}^2 - \text{C E}^2)} =$

$$\sqrt{\left(\text{AC}^2 - \frac{1+\sqrt{5}}{2\sqrt{3}}\right)^2} = \text{A C}\sqrt{\left(1-\frac{3+\sqrt{5}}{6}\right)} = \text{A C}\sqrt{\frac{3-\sqrt{5}}{6}}.$$



1. If the lineal side of the dodecaedron be 1, what is its solidity?

Here $A = 1$, consequently $5\,A^3\sqrt{\dfrac{47+21\sqrt{5}}{40}} =$ 7·66311896 is the content.

Now, the chord of an arc being a mean proportional between its versed sine and the diameter, A E the versed sine whose chord is A C, and its diameter equal to that of the circumscribed sphere; we have

$$AC^2 \div 2\,AE = AC^2 \div 2\,AC\sqrt{\frac{3-\sqrt{5}}{6}} = \tfrac{1}{2}\,AC\sqrt{\frac{6}{3-\sqrt{5}}} = \tfrac{1}{4}\,AC\sqrt{6}\times\frac{3+\sqrt{5}}{9-5} = \tfrac{1}{4}\,AC\times 3\times\frac{6+2\sqrt{5}}{4} = AC\times\frac{1+\sqrt{5}}{4}\times\sqrt{3} = \frac{\sqrt{3}+\sqrt{15}}{4}\,AC = R,$$ the radius of the circumscribed sphere.

Again, the angle A F C contains 72 degrees, whose sine is $\tfrac{1}{4}\sqrt{(10+2\sqrt{5})}$. The angle A C F is 54 degrees, whose sine is $\dfrac{1+\sqrt{5}}{4}$.

Hence $\sqrt{(10+2\sqrt{5})} : 1+\sqrt{5} :: AC : AF = \dfrac{1+\sqrt{5}}{\sqrt{(10+2\sqrt{5})}}\times AC = AC\sqrt{\dfrac{5+\sqrt{5}}{10}}$. Now it is obvious that the radius of the circumscribed sphere is the hypothenuse of a right-angled triangle, whose two legs are A F and the radius of the inscribed sphere; hence

2. The side of a regular dodecaedron is 12 inches, how many cubic inches does it contain?

Ans. 13241·8694592 inches.

PROBLEM V.

To find the solidity of an icosaedron.

RULE. To 7 add three times the square root of 5, take half the sum, multiply the square root of this half sum by $\frac{5}{6}$ of the cube of the lineal side, and the product will be the solidity.*

---

we have $\sqrt{(R^2 - AF^2)} = \sqrt{\left[\left(\frac{\sqrt{3}+\sqrt{15}}{4}AC\right) - \frac{5+\sqrt{5}}{10}AC^2\right]}$ $= AC\sqrt{\left(\frac{18+6\sqrt{5}}{16} - \frac{5+\sqrt{5}}{10}\right)} = AC\sqrt{\frac{25+11\sqrt{5}}{40}} = r$, the radius of the inscribed sphere.

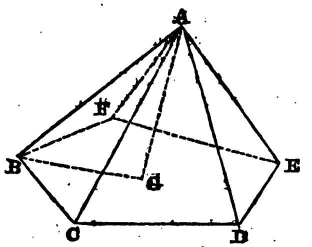
Now, the solidity of any regular solid is equal to the surface multiplied by $\frac{1}{3}$ of the radius of the inscribed sphere, and the surface, as will be shown hereafter, is equal to $15A^2\sqrt{\frac{5+2\sqrt{5}}{5}}$; therefore,

$$B \times \tfrac{1}{3}r = 15A^2\sqrt{\frac{5+2\sqrt{5}}{5}} \times \tfrac{1}{3}A\sqrt{\frac{25+11\sqrt{5}}{40}} = 5A^3 \times$$

$\sqrt{\frac{47+21\sqrt{5}}{40}} = C$ the solidity, A being the lineal side, B the surface, and C the solidity.

* *Demonstration.* Let A be the solid angle of the icosaedron, formed by 5 triangles whose bases form the pentagon B C D E F, on the centre of which let fall the perpendicular A C, join B G.

In one of the steps of the last demonstration, it was shown that $BG = AB\sqrt{\frac{5+\sqrt{5}}{10}}$, and the radius of the circle circumscribing one of the faces A B C, of the solid $= A\sqrt{\frac{1}{3}}$. But the radius of the circumscribing sphere is $R = \frac{AB^2}{2AG} = \frac{AB^2}{2\sqrt{(AB^2 - BG^2)}} =$ $\frac{AB^2}{2\sqrt{(AB^2 - \frac{5+\sqrt{5}}{10}AB^2)}} = \frac{AB}{2\sqrt{(1 - \frac{5+\sqrt{5}}{10})}} = \frac{AB}{2\sqrt{\frac{5-\sqrt{5}}{10}}} =$



1. What is the solidity of an icosaedron, whose lineal side is 1?

Let the side be denoted by A. Then A = 1, and consequently

$$\tfrac{5}{6} A^3 \sqrt{\frac{7+3\sqrt{5}}{2}} = \tfrac{5}{6} \sqrt{\frac{7+3\sqrt{5}}{2}} = 2{\cdot}18169499,$$

the content.

2. What is the solidity of an icosaedron, whose lineal side is 12 feet? *Ans.* 3769·96896 feet.

Note. The following table may be collected from the examples

$$\tfrac{1}{2} AB \sqrt{\frac{10}{5-\sqrt{5}}} = \tfrac{1}{2} AB \sqrt{\left(\frac{10}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}}\right)} =$$

$$\tfrac{1}{2} AB \sqrt{\frac{10 \times (5+\sqrt{5})}{25-5}} = AB \sqrt{\frac{5+\sqrt{5}}{8}}.$$

Now, R is the hypothenuse of a right-angled triangle of which the one leg is Q ($= AB\sqrt{\tfrac{1}{3}}$) the radius of the circle circumscribing one of the faces A B C, and the other the radius r of the inscribed sphere.

$$\text{Hence } r = \sqrt{(R^2 - Q^2)} = \sqrt{\left(\frac{5+\sqrt{5}}{8} AB^2 - \tfrac{1}{3} AB^2\right)} =$$

$$AB \sqrt{\frac{15+3\sqrt{5}-8}{8 \times 3}} = AB \sqrt{\frac{7+3\sqrt{5}}{24}}.$$

If the whole surface be denoted by B, and the solidity by S, we shall have

$$S = \tfrac{1}{3} r B = \tfrac{1}{3} AB \sqrt{\frac{7+3\sqrt{5}}{24}} \times 5 AB^2 \sqrt{3} = \tfrac{5}{3} AB^3 \sqrt{\frac{7+3\sqrt{5}}{8}}$$

$= \tfrac{5}{6} AB^3 \sqrt{\frac{7+3\sqrt{5}}{2}}$, which is the rule.

given in the foregoing rules, each of which has been demonstrated under its particular head. It has also been demonstrated that the cube of the lineal side of any regular solid multiplied by the tabular number corresponding to the figure, will give its contents. It is particularly recommended to the pupil, to employ the general rule given in Problem I. whenever the contents of any of the five regular bodies is required.

TABLE IV.

Showing the solidity of the five regular bodies, the length of a side in each being 1.

No. of sides.	Names.	Solidity.
4	Tetraedron ..	·1178511
8	Octaedron ..	·4714045
20	Icosaedron ..	2·1816950
12	Dodecaedron ..	7·6631189
6	Hexaedron ..	1·

PROBLEM VI.

To find the surface of a tetraedron.

RULE I. Multiply the square of the lineal side by the square root of 3, and the product will be the whole surface.*

The following rule is general for finding the superficies of any of the five regular bodies.

II. Multiply the square of the length of a side of the body, by the tabular area corresponding to the figure, and the product will be the surface of the body.

1. If the side of a tetraedron be 1, what is its surface?

---

* *Demonstration.* The area of an equi-lateral triangle (Problem VI, Section II.) is $\frac{A^2}{4}\sqrt{3}$, A being one of the sides; then, the area of the four faces will be $A^2\sqrt{3}$, which is the first rule. The reason of the second rule is obvious from the property, that similar surfaces *are to each* other as the squares of their like sides.

Here $1^2 \times \sqrt{3} = \sqrt{3} = 1{\cdot}7320508 =$ the whole surface.

2. The side of a tetraedron is 12, what is its surface?
Ans. 249·4153152.

PROBLEM VII.

To find the surface of a hexaedron.

RULE. Square the side and multiply it by 6, and the product will be the surface.*

1. If the side be 1, what is the surface of a hexaedron?
$1^2 \times 6 = 6$ the whole surface.

2. If the side be 4, what is the surface of a hexaedron?
Ans. 96.

PROBLEM VIII.

To find the surface of an octaedron.

RULE. Multiply the square of the side by the square root of 3, and double the product will be the surface.†

1. If the side of an octaedron be 1, what is its surface?
$2 \times 1^2 \sqrt{3} = 2\sqrt{3} = 3{\cdot}4641016 =$ the whole surface.

2. If the side of an octaedron be 12, what is its superficies? *Ans.* 498·8306304.

3. If the side of an octaedron be 4, what is its surface?
Ans. 55·4256256.

---

* *Demonstration.* The hexaedron is composed of six square faces, the area of each being A^2, (A being the side,) therefore $6\,A^2$ is the whole surface.

† By Problem VI. Section II. the area of one of the faces is $\frac{A^2}{4}\sqrt{3}$, (A being a side,) therefore the surface of the 8 faces of the octaedron is $\frac{A^2}{4}\sqrt{3} \times 8 = 2\,A^2\sqrt{3}$, which the rule.

PROBLEM IX.

To find the superficies of a dodecaedron.

RULE. To 1 add $\frac{2}{5}$ of the root of 5; multiply the root of the sum by 15 times the square of the lineal side, and the product will be the surface.*

1. If the lineal side be 1, what is the surface of a regular dodecaedron?

Here $1^2 \times 15\sqrt{(1+\frac{2}{5}\sqrt{5})} = 15\sqrt{(1+\frac{2}{5}\sqrt{5})} = 20{\cdot}6457788807$, the surface.

2. What is the surface of a dodecaedron, whose lineal side is 2? *Ans.* 82·58292.

PROBLEM X.

To find the superficies of an icosaedron.

RULE. Multiply five times the square of the lineal side by the square root of 3, and the product will be the surface.†

---

* *Demonstration.* In the pentagon, the angle made by its side with the radius of the circumscribing circle is 54 degrees, whose sine is $\frac{\sqrt{5}+1}{4}$, and its co-sine $\frac{\sqrt{(10-2\sqrt{5})}}{4}$. (See *Trigonometry.*) But the tangent is equal to the sine divided by the co-sine; therefore the tangent of 54°, or $t = \frac{\sqrt{5}+1}{\sqrt{(10-2\sqrt{5})}} = \frac{6+2\sqrt{5}}{10-2\sqrt{5}} = \sqrt{(1+\frac{2}{5}\times\sqrt{5})}$; hence the area of the pentagon is $\frac{5}{4}\sqrt{(1+\frac{2}{5}\sqrt{5})}$, which being multiplied by 12, will give the whole surface, that is $12\times\frac{5}{4}\sqrt{(1+\frac{2}{5}\sqrt{5})} = 5\sqrt{(1+\frac{2}{5}\sqrt{5})}$ which is the rule.

† *Demonstration.* By Problem VI. Section II., the area of one of the faces is $\frac{A^2}{4}\sqrt{3}$, (A being one of the sides); but the figure has 20 such faces; therefore $20\times\frac{A^2}{4}\sqrt{3} = 5A^2\sqrt{3}$ is the surface of the whole solid.

1. The side of an icosaedron is 1, what is its surface?

$$5 \times 1^2 \times \sqrt{3} = 5\sqrt{3} = 8{\cdot}66025403.$$

2. What is the surface of an icosaedron, whose side is 2?
Ans. 34·641.

3. What is the surface of an icosaedron, whose side is 3?
Ans. 77·9423.

Note. It is particularly recommended to employ the general rule given in Problem I., in practice, in preference to any other. The particular rules given for each solid are introduced merely to find the tabular numbers, by which the pupil is to work.

From the examples given in the preceding rules, in which the lineal side of each regular solid is 1, the following tabular numbers may be collected.

TABLE V.

Shewing the surfaces of the five regular bodies, when the lineal side is 1.

No. of sides.	Names.	Surface.
4	Tetraedron ..	1·7320508
6	Hexaedron ..	6·0000000
8	Octaedron ..	3·4641016
12	Dodecaedron ..	20·6457788
20	Icosaedron ..	8·6602540

This Table may be calculated from Table II., by multiplying the tabular numbers there corresponding to the faces of the regular bodies, by the number of such faces forming the solid. Thus 4 times the tabular number corresponding to an equi-lateral triangle will be the tabular number corresponding to the tetraedron; 6 times the tabular number answering to a square will be the tabular number answering to the hexaedron; 8 times the tabular number answering to the triangle will be the tabular number that answers the octaedron; and so of the rest.

It is usual in works on Mensuration to treat the preceding and following Sections under one head, but for the sake of distinction, and for other reasons which may be best seen in the operations, they are here separated, and treated under distinct heads.

SURFACES OF SOLIDS.

SECTION VII.

PROBLEM I.

To find the surface of a cube.

RULE. Square the side, and multiply this square by 6, the product will be its surface.

Let A be the side, then $6A^2$ is the surface.

The reason of this rule is obvious; for A^2 is the area of one of the faces; therefore $6A^2$ will be the area of the six faces.

1. What is the surface of a cube, whose side is 1?

Here $6A^2 = 6 \times 1^2 = 6$, the surface required.

2. What is the surface of a cube, whose side is 10?

Ans. 600.

PROBLEM II.

To find the surface of a parallelopipedon.

RULE. Find the area of the sides and ends, and their sum will be the surface.

1. What is the surface of a parallelopipedon, whose length is 10 feet, breadth 4, and depth 2? *Ans.* 120 feet.

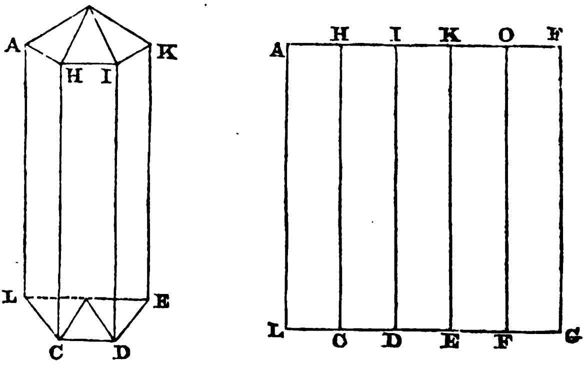
$$
\begin{array}{rl}
10 \times 4 = 40 = & \text{the area of one face.} \\
10 \times 4 = 40 = & \text{the area of its opposite face.} \\
10 \times 2 = 20 = & \text{the area of one face.} \\
10 \times 2 = 20 = & \text{the area of its opposite face.} \\
4 \times 2 = 8 = & \text{the area of one end.} \\
4 \times 2 = 8 = & \text{the area of its opposite end.} \\
\hline
136 = & \text{the surface of the whole solid.}
\end{array}
$$

2. The length of a parallelopipedon is 5, breadth 4, and depth 3, what is its surface? *Ans.* 94.

PROBLEM III.

To find the surface of a prism.

RULE. Multiply the circumference of the end of the solid by its length, to the product add the area of the two ends, and the sum will be the surface.*



1. If the side H I of the pentagon be 25 feet, and height I D 10, what is its surface?

$$25 \times 5 = 125, \text{ the perimeter};$$

$$\text{Then } 125 \times 10 = 1250 = \text{the upright surface};$$

$$25^2 \times 1{\cdot}720477 = 1075{\cdot}298125 = \text{the area of one end};$$

* *Demonstration.* If we conceive the pentagonal prism C D E, &c. to be formed of pasteboard and unfolded, the convex surface of it will, when extended, form a parallelogram A F G L, whose altitude is equal to that of the prism, and base L G equal to the circumference, or perimeter of the pentagon; but the area of the parallelogram is, A L $\times$ L G; therefore the convex surface of the prism is $h \times p$, h being its height, and p (= L G) its circumference or perimeter, to which the areas of both ends are to be added to find the surface of the entire prism, which is the rule.

And $1075{\cdot}298125 \times 2 = 2150{\cdot}596250 =$ the area of both ends;

Then $2150{\cdot}596250 + 1250 = 3400{\cdot}59625 =$ the entire surface.

2. What is the surface of a cylinder, whose diameter is 27 inches, and height 16 feet? *Ans.* 129·00195 feet.

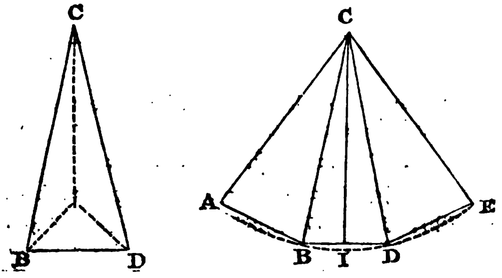
Note. The same rule answers for the cylinder, and every kind of prism.

PROBLEM VI.

To find the surface of a pyramid.

RULE. Multiply the slant height by half the circumference of the base, and the product will be the surface of the sides, to which add the area of the base for the whole surface.*

Note. The slant height of a pyramid is the perpendicular distance from the vertex to the middle of one of the sides, and the perpendicular height is a straight line drawn from the vertex to the middle of the base.



* *Demonstration.* If we conceive a triangular pyramid made of pasteboard to be unfolded, it is obvious that the surface of its height will be equal to that of C A B D E which is composed of three equal triangles, viz. the three faces of the pyramid; but the area of C B D is $CI \times \frac{1}{2} BD$; therefore the area of the three faces is $CI \times \frac{1}{2}(AB + BD + DE) = CI \times \frac{1}{2}p$, p being the perimeter; hence $CI \times \frac{1}{2}p$, together with the area of the base, is the whole surface.

1. The slant height of a triangular pyramid is 10 feet, and each side of the base is 1; what is its surface?

Here $\frac{1}{2}p \times CI = \frac{3}{2} \times 10 = 15 =$ the upright surface.

And $\cdot 433013 =$ the area of the base.

Then $15 + \cdot 433013 = 15\cdot 433013 =$ the entire surface.

2. The perpendicular height of a heptagonal pyramid is 13·5 feet, and each side of the base 15 inches; required its surface? *Ans.* 65·0128 feet.

PROBLEM V.

To find the surface of a cone.

RULE. Multiply the slant height by half the circumference of the base, and the product, with the area of the base, will be the whole surface.*

1. What is the surface of a cone whose side is 20, and the circumference of its base 9?

Here $20 \times \frac{9}{2} = 90 =$ the convex surface.

$9^2 \times \cdot 07958 = 6\cdot 44598 =$ the area of the base.

Then $90 + 6\cdot 44598 = 96\cdot 44598 =$ the whole surface.

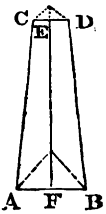
2. The perpendicular height of a cone is 10·5 feet, and the circumference of its base is 9 feet; what is its superficies? *Ans.* 54·1336 feet.

* *Demonstration.* If a circular sector be described on paper, so that its radius shall be equal to the side of the cone, and its arc equal to the circumference of the base, this sector can be rolled round the cone, so as to cover it exactly; but the area of this sector is found by multiplying the radius of the sector by half the arc; therefore the convex surface of the cone is found by multiplying the slant height by half the circumference of the base, which, with the area of the base, is the whole surface.

PROBLEM VI.

To find the superficies of the frustrum of a pyramid.

RULE. Add the perimeters of the two ends together, and multiply half the sum by the slant height, the product will be the upright surface; to which add the areas of both ends, and the sum will be the whole surface.*



1. What is the superficies of the frustrum of a square pyramid, each side of the greater base A B being 10 inches, and each side of the less base C D 4 inches, and slant height 20 inches?

Here $10 \times 4 = 40$ the perimeter of the greater base.

And $4 \times 4 = 16$ the perimeter of the less end.

Sum 56, the half of which is 28.

Then $28 \times 20 = 560 =$ the upright surface.

$10 \times 10 = 100 =$ the area of the greater base.

$4 \times 4 = 16 =$ the area of the less end.

Hence $560 + 100 + 16 = 676 =$ the whole surface.

2. What is the superficies of the frustrum of an octogonal pyramid, each side of the greater base being 9 inches, each side of the less base 5 inches, and the length 10·5 feet?

Ans. 52·590223 feet.

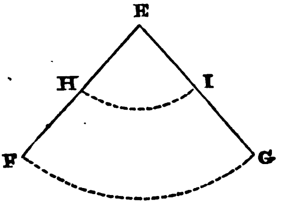
---

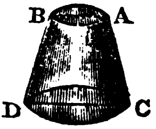
* *Demonstration.* Let A B D C be one of the faces of the frustrum of a pyramid, E F, joining the middle of A B, and C D, is the slant height. Now, it is obvious that when the ends are regular polygons, the upright surface will consist of as many trapezoids, each equal to A B D C, as there are sides in the polygon, the common height being E F; but the area of the face A B D C is $\frac{AB + CD}{2} \times EF$; therefore the area of the whole upright surface is $\frac{P + p}{2} \times EF$. (P and p being the perimeters of the two ends of the frustrum,) to which add *the areas of both* ends, for the entire surface.

PROBLEM VII.

To find the superficies of the frustrum of a cone.

RULE. Add the perimeters of both ends together, and multiply half the sum by the slant height, to which add the areas of both ends, for the whole superficies.*





1. If the diameters of the two ends C D and A B are 7 and 3, and the slant height D B 9, what is the whole surface of the frustrum A B D C?

$\frac{7+3}{2} \times 3{\cdot}1416 \times 9 = 141{\cdot}372$, the convex surface.

$7 \times 7 \times {\cdot}7854 = 28{\cdot}6846$, the area of the base C D.
$3 \times 3 \times {\cdot}7854 = 7{\cdot}0686$, the area of the end A B.

Then $141{\cdot}372 + 35{\cdot}7532 = 177{\cdot}1252 =$ the whole surface of the frustrum.

---

* *Demonstration.* If a part of the sector E F G, viz. H F G I, having H F $=$ B D, be rolled round the frustrum A B D C, so as to cover it exactly, it is evident that the area of the envelope H F G I will be equal to the convex surface of the cone A B D C. But the area of the envelope is $\frac{\text{HI}+\text{FG}}{2} \times$ F H, and H I is equal to the perimeter of the less end A B of the cone, and F G equal the perimeter of the greater base C D; therefore $\frac{P+p}{2} \times$ B D is the convex surface of the cone, P being the perimeter of the greater base, and p that of the less; therefore $\frac{P+p}{2} \times$ B D, together with the areas of both ends, will be the entire surface.

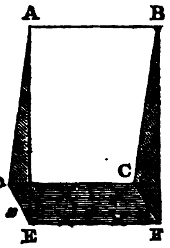
2. What is the superficies of the frustrum of a cone, whose greater diameter is 18 inches, and less diameter 9 inches, and the slant height 171·0592 inches?

Ans. 7572·98136672.

PROBLEM VIII.

To find the superficies of a wedge.

RULE. Find the area of the back, which is a right-angled parallelogram; find the areas of both ends, which are triangles; and also of both sides, which are parallelograms; and add all the separate areas together for the whole surface.*



1. The back of a wedge is 10 inches long, and 2 inches broad, each of its faces is 10 inches from the edge to the back; required its whole surface?

$10 \times 2 = 20 =$ the area of the back.

$10 \times 10 \times 2 = 200$ the areas of both faces.

$\sqrt{(A\,E^2 - E\,x^2)} = \sqrt{(100 - 1)} = 9{\cdot}949 = A\,x$; then

$9{\cdot}949 \times 2 = 19{\cdot}998 =$ areas of both ends.

Hence $200 + 20 + 19{\cdot}898 = 239{\cdot}898 =$ the whole surface of the wedge.

2. The back of a wedge is 20 inches long, and 2 inches broad; each of its faces is 10 inches from the back to the edge; what is its whole surface? *Ans.* 459·898.

---

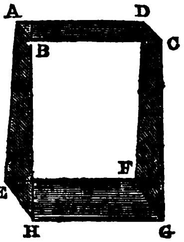
* *Demonstration.* The back E D C F is a parallelogram, having each of its angles right, its area, therefore, is $C\,D \times C\,F$. The faces A D C B and A E F B are rectangular parallelograms, and their areas are $2 \times A\,D \times D\,C$. The ends A D E and B C F are equal triangles, whose areas are $A\,x \times D\,E$. Then $C\,D \times C\,F + 2\,A\,D \times D\,C + A\,x \times D\,E =$ the whole surface.

PROBLEM IX.

To find the area of the frustrum of a wedge.

RULE. Find the areas of the back and top section of the two faces, and of the two ends, and the sum of all the separate results will be the whole surface.*

1. The length and breadth of the back are 10 and 2 inches, the length and breadth of the upper section are 10 and 1 inches, the length of the edge from the back to the upper section is 10 inches; required the whole surface?



$10 \times 2 = 20 =$ the area of the back.
$10 \times 1 = 10 =$ the area of the upper section.

$10 \times 10 \times 2 = 200 =$ the areas of both faces.

$\frac{2-1}{2} = \frac{1}{2} = \cdot 5$, and $\sqrt{(100 - \cdot 25)}\ 9\cdot 98 = xy$.

Then $(2+1) \times 9\cdot 98 = 29\cdot 94 =$ areas of both ends.
Hence $20 + 10 + 200 + 29\cdot 94 = 259\cdot 94$ inches, the *Ans*.

2. The length and breadth of the back are 10 and 4, the length and breadth of the upper section are 5 and 2, and the length of each of the faces is 20; required the whole superficies? *Ans*. 467·495.

PROBLEM X.

To find the surface of a globe or sphere.

RULE. Multiply the diameter of the sphere by its circumference, and the product will be its convex surface.†

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* This rule is evident from the last. The back E H G F is a rectangular parallelogram, the upper section A B C D is another, the faces A E F D and B H G C are rectangular parallelograms also, and the ends D F G C and A E H B are two trapezoids; then the sum of all the separate areas is the surface of the whole frustrum.

† *Demonstration.* It is proved in Dr. Lardner's *Euclid*, Prop. X.

1. What is the surface of a globe, whose diameter is 24 inches?

$$24 \times 3{\cdot}1416 = 75{\cdot}3984\text{; then}$$
$$75{\cdot}3984 \times 24 = 1809{\cdot}5616 \text{ inches, the answer.}$$

2. What is the surface of the earth, its diameter being 7957¾, and circumference 25000 miles?

Ans. 198943750 square miles.

PROBLEM XI.

To find the convex surface of any segment, or zone of a sphere.

RULE. Multiply the circumference of the whole sphere by the height of the segment, or zone, and the product will be the convex surface.*

1. If the diameter of the earth be 7970 miles, the height of the frigid zone will be 252·361283 miles; what is its surface?

Here $7970 \times 3{\cdot}1416 =$ the circumference; then
$$7970 \times 3{\cdot}1416 \times 252{\cdot}361283 = 6318761{\cdot}107182216$$
miles.

2. If the diameter of the earth be 7970 miles, the height

---

that the surface of a sphere is equal to that of the circumscribing cylinder; but the cylindrical surface is equal to the circumference of its base, which is equal to that of the sphere, multiplied by its altitude, which is equal to a diameter of the sphere. Therefore the surface of the sphere is equal to its circumference multiplied by its diameter.

* *Demonstration.* It is proved in Dr. Lardner's *Euclid*, Prop. XI., that any plane intersecting a sphere and its circumscribing cylinder parallel to the base of the cylinder, divides the spherical and cylindrical surfaces into parts which are equal to each other. Therefore if two such planes be drawn, the spherical and cylindrical surfaces which they include will be the difference between the equal spherical and cylindrical surfaces which they cut off towards either of the bases of the cylinder, and therefore those differences are equal.

But the surface of the cylindrical segment is found by multiplying its circumference, which is equal to that of the sphere, by its length, *which is equal either to the distance between the parallel planes, or to the height of the spherical segment*; hence the reason of the rule.

of the temperate zone will be 2143·6235535 miles; what is its surface? *Ans.* 53673229·812734532 miles.

3. If the diameter of the earth be 7970 miles, the height of the torid zone will be 3178·030327 miles; what is its surface? *Ans.* 79573277·600166504 miles.

Note. By adding the surfaces of both frigid zones and both temperate zones, to the surface of the torid zone, the sum 199557259·44, is the surface of the earth in square miles.

4. The diameter of a sphere is 3, the height of the segment 1; what is its convex surface? *Ans.* 9·4248.

5. The circumference of a sphere is 33, the height of the segment is 4; what is its convex surface? *Ans.* 132.

PROBLEM XII.

To find the surface of a cylinder.

RULE. Multiply the circumference by the length, and the product will be the convex surface; to which add the area of the two ends, and the sum will be the surface of the entire solid.*

1. What is the entire surface of a cylinder, whose length is 10 feet, and its diameter 5 feet?

$$\begin{array}{r} 3{\cdot}1416 \\ 5 \\ \hline 15{\cdot}7080 \end{array}$$

15·7080, then $15{\cdot}708 \times 10 = 157{\cdot}08 =$ the convex surface.

$5 \times 5 \times {\cdot}7854 =$ the area of the base; then

$2 \times 5 \times 5 \times {\cdot}7854 = 50 \times {\cdot}7854 = 39{\cdot}2700 =$ the area of both bases; then

$157{\cdot}08 + 39{\cdot}27 = 196{\cdot}35$, the answer.

2. Required the superficial content of a cylinder, whose diameter is 21·5 inches, and height 16 feet? *Ans.* 95·1 ft.

---

* *Demonstration.* The envelope of a cylinder is a parallelogram whose sides are evidently the height and circumference of the cylinder; therefore the area of such a parallelogram is equal to the convex surface of the cylinder, to which the area of the two ends being added, the sum will give the entire surface of the solid.

3. What is the surface of a cylinder whose diameter is 20·75 inches, and its length 55 inches? *Ans.* 29.595 feet.

PROBLEM XIII.

To find the superficies of a circular cylinder.

RULE. Add the inner diameter to the thickness of the ring, multiply the sum by the thickness, and that product by 9·8696 for the superficies.*

1. The thickness A C of a cylindrical ring is 2 inches, the diameter C D 5 inches; required its superficial content?

Here $(2 + 5) \times 2 = 14$; then $14 \times 9{\cdot}8696 = 138{\cdot}1744$, square inches.

---

* *Demonstration.* See Fig. Prob. XXVIII. $AC + CD = mn$, then the mean length of the cylinder is $(AC + CD) \times 3{\cdot}1416$; but the circumference of a section A C is $AC \times 3{\cdot}1416$, then by the last Problem the surface is $(AC + CD) \times AC \times 3{\cdot}1416 \times 3{\cdot}1416 = (AC + CD) \times AC \times 9{\cdot}8696$, which is the rule.

CARPENTERS' RULE OR SLIDE & RULE.

SECTION VIII.

This instrument is sometimes called the sliding rule, and is used in measuring timber, and Artificers' works. By it likewise dimensions are taken, and contents computed.

It consists of two equal pieces of box, each one foot long, connected by a folding joint.

One face of the rule is divided into inches and half-quarters or eighths. On the same side or face are several plane scales divided by diagonal lines into twelfths; these are chiefly used in planning dimensions which are taken in feet and inches. The edge of the rule is divided decimally; that is, each foot is divided into 10 equal parts, and each of those again into 10 equal parts. By means of this last scale, dimensions are taken in feet, tenths, and hundredths; and then multiplied as common decimal numbers.

In one of these equal pieces, there is a slider on which are marked the two letters B, C; on the same face are marked the letters A, D. The same numbers serve for both these two middle lines, the one being above the numbers, and the other below.

Three of these lines, viz. A, B, C, are called double lines, as they proceed from 1 to 10 twice over. These three lines are exactly alike both in division and numbers, and are numbered from the left-hand towards the right 1, 2, 3, 4, 5, 6, 7, 8, 9, 1 which stands in the middle; the numbers then go on, 2, 3, 4, 5, 6, 7, 8, 9, 10, which stands at the right-hand end of the rule.

These four lines are logarithmic ones; the lower line D, is a single one, proceeding from 4 to 40, and is called the girt line from its use in finding the contents of timber.

Upon it are also marked W G at 17·15, A G at 18·95, and I G at 18·8. These are the wine, ale, and imperial gauge points.

On this face is a table of the value of a load, or 50 cubic feet, of timber, at all prices from 6 pence to 2 shillings per foot.

To ascertain the values of the figures on the rule, which have no determinate value of their own, but depend upon the value set on the unit at the left-hand of that part of the rule marked 1, 2, 3, &c.; if the first unit be called 1, the 1 in the middle will be 10, the other figures that follow will be 20, 30, 40, &c., and the 10 at the right-hand end will be 100. If the left-hand unit be called 10, the one in the middle will be 100, and the following figures will be 200, 300, 400, 500, &c.; and the 10 at the right-hand end will be 1000. If the 1 at the left-hand end be called 100, the middle 1 will be 1000, and the following figures will be 2000, 3000, 4000, &c., and the 10 at the right-hand will be 10,000. From this it appears, that the values of all the figures depend upon the value set on the first unit.

The use of the double line A, B, is to find a fourth proportional, and also to find the areas of plane figures.

The use of the several lines described here is best learned in practice.

If the rule be unfolded, and the slider moved out of the grove, the back part of it will be seen divided like the edge of the rule, all measuring 3 feet in length.

Some rules have other scales and tables delineated upon them; such as a table of board measure, one of timber measure, another for showing what length for any breadth will make a foot square. There is also a line showing what length for any thickness will make a solid foot.

THE USE OF THE SLIDING RULE.

PROBLEM I.

To multiply numbers together.

Set 1 on B to the multiplier on A; then against the multiplicand on B, stands the product on A.

1. Multiply 12 and 18 together.

Set 1 on B, to 12 on A; then against 18 on B, stands the product 216 on A.

2. Multiply 36 by 22.

Set 1 on B, to 36 on A; then as 22 on B goes beyond the rule, look for 2·2 on B, and against it on A stands 79·2; but as the real multiplier was divided by 10, the product 79·2 must be multiplied by 10, which is effected by taking away the decimal point, leaving the product 792.

PROBLEM II.

To divide one number by another.

Set the divisor on A, to 1 on B; then against the dividend on A, stands the quotient on B.

1. Divide 11 into 330.

Set the divisor 11 on A, to 1 on B; then against the dividend 330 on A, stands the quotient 30 on B.

2. Divide 7680 by 24.

Set 24 on A, to 1 on B; then because 7680 goes beyond the rule on A, look for 768, (the tenth of 7680,) on A, and against it stands 32 on B; but as the tenth of the dividend was taken that the number should fall within the compass of the scale A, the quotient 32 must be multiplied by 10, which gives 320 for the answer.

PROBLEM III.

To square any number.

Set 1 upon C, to 10 upon D; then if you call the 10 upon D, I, the 1 on C will be 10; if you call the 10 on D, 10, then the 1 on C will be 100; if you call the 10 on D, 100, then the 1 on C will be 1000; this being understood, you will observe that against every number on D, stands its square on C.

1. What is the square on 25?

Proceeding according to the above directions, 625 stands against 25, 900 against 30, 144 against 12, 400 against 20, &c.

PROBLEM IV.

To extract the square root of a number.

Set 1 or 100, &c. on C, to 1 or 10, &c. on D; then against every number found on C, stands its root on D.

1. What is the square root of 529?

Proceeding according to the above directions, opposite 529 stands 23; opposite 1600 stands 40, and so on.

PROBLEM V.

To find a mean proportional between two numbers, as 9 and 25.

Set the number 9 on C, to the same 9 on D; then against 25 on C, stands 15 on D, the required mean proportional.

The reason of this may be seen from the proportion, viz. 9 : 15 :: 15 : 25.

1. What is the mean proportional between 29 and 430?

Set one number 29 on C, to the same on D; then against the other number 430 on C, stands the required mean proportional 112, on D nearly.

PROBLEM VI.

To find a third proportional to two numbers, as 21 and 32.

Set the first 21 on B, to the second 32 on A; then against the second 32 on B, stands 48·8 on A, which is the required third proportional.

PROBLEM VII.

To find a fourth proportional to three given numbers.

Set the first term on B, to the second on A; then against the third term on B, stands the fourth on A.

If either of the middle numbers fall beyond the line, take one-tenth part of that number, and increase the fourth number found ten times.

1. Find a fourth proportional to 12, 28, and 114?

Set the first term, 12, on B, to the second term, 28, on A; then against the third term 114 on B, stands 266 on A, which is the answer.

TIMBER MEASURE.

PROBLEM I.

To find the superficial content of a board or plank.

RULE. Multiply the length by the breadth, and the product will be the area.

Note. When the plank is broader at one end than at the other, add both ends together, and take half the sum for a mean breadth.*

---

* This rule is already demonstrated.

BY THE CARPENTERS' RULE.

Set 12 on B, to the breadth in inches on A, then against the length in feet, on B, will be found the superficies on A, in feet.

1. If a board be 12 feet 6 inches long, and 2 feet 3 inches broad, how many feet are contained in it?

12 : 6	12·5
2 : 3	2·25
25 : 0	625
3 : 1·6	250
	250
28 : 1·6 *Ans.*	
	28·125 *Ans.*

BY THE CARPENTERS' RULE.

As 12 on B : 27 on A : : 12·5 on B : 28·125 on A.

2. What is the value of a board whose length is 8 feet 6 inches, and breadth 1 foot 3 inches; at 5*d.* per foot? *Ans.* 4*s.* 5*d.*

3. What is the value of a board whose length is 12 feet 9 inches, and breadth 1 foot 3 inches; at 5*d.* per foot? *Ans.* 6*s.* 7½*d.*

4. What is the value of a plank whose breadth at one end is 2 feet, and at the other end 4 feet, at 6*d.* per foot, the length being 12 feet? *Ans.* 18*s.*

5. How many square feet in a board, whose breadth at one end is 15 inches, and at the other 17 inches, the length being 6 feet? *Ans.* 8.

6. How many square feet in a plank, whose length is 20 feet, and mean breadth 3 feet 3 inches? *Ans.* 65.

PROBLEM II.

To find the solid content of squared or four-sided timber.

RULE. Take half the sum of the breadth and depth *in the* middle, (that is, the quarter girt,) square this

half sum, and multiply it by the length for the solid content.*

BY THE CARPENTERS' RULE.

As 12 on D : length on C :: quarter girt on D : the solid content on C.

1. If a piece of squared timber be 3 feet 9 inches broad, 2 feet 7 inches deep, and 20 feet long; how many solid feet contained therein?

```
   3 : 9
   2 : 7
 -------
 2)6 : 4
 -------
   3 : 2   quarter girt.
   3 : 2
   -----
   9 : 6
     6:4
  -------
  10 : 0:4  square of the quarter girt.
       20   length of the piece.
 ---------
 200 : 6:8  solid content.
```

BY THE CARPENTERS' RULE.

As 12 on D : 20 on C :: 38 on D : 200½ on C.

2. A squared piece of timber is 15 inches broad, 15 inches deep, and 18 feet long; how many solid feet does it contain?

Ans. 28⅛ feet, which is the accurate content, as the breadth and depth are equal.

3. What is the solid content of a piece of timber, whose breadth is 16 inches, depth 12 inches, and length 12 feet?

Ans. 16 feet.

---

* This rule, which is generally employed in practice, is far from being correct, when the mean breadth and depth differ materially from each other, and the timber does not taper.

RULE II. Multiply the breadth in the middle by the depth in the middle, and that product by the length, for the solidity.*

4. The length of a piece of timber is 18 feet 6 inches; the breadths at the greater and less end 1 foot 6 inches, and 1 foot 3 inches, and the thickness at the greater and less end 1 foot 3 inches, and 1 foot; what is the solid content?

```
  1·5                      1·25
  1·25                     1·
 ------                   ------
2)2·75                   2)2·25
 ------                   ------
 1·375 mean breadth.      1·125 mean depth.
             1·125 mean depth.
             1·375 mean breadth.
             --------
             1·546875
                18·5 length.
            -----------
            28·6171875 solid content.
```

BY THE SLIDING RULE.

	B		A		B		A	
As	1	:	$13\frac{1}{2}$	::	$16\frac{1}{2}$	:	223	the mean square.
	C		D		C		D	
As	1	:	1	::	223	:	14·9	quarter girt.
	C		D		D		C	
As	$18\frac{1}{2}$	:	12	::	14·9	:	28·6	the content.

Note. When the piece to be measured tapers regularly from one end to the other, either take the mean breadth and depth in the middle, or take the dimensions at both ends, and half their sum for the mean dimension. This, however, though very easy in practice, is but a very imperfect approximation.

When the piece to be measured does not taper regularly, but is thick in some parts and small in others, in this case take several

* This rule is correct when the timber does not taper; but when the timber tapers considerably, and the breadth and depth nearly equal, the rule is very erroneous. The measurer, therefore, ought to consider the shape of the timber he is about to measure before he applies either of the above rules.

dimensions; add them all together, and divide their sum by the number of dimensions, so taken; and use the quotient as the mean dimension.

RULE III. Multiply the sum of the breadths of the two ends by the sum of the depths, to which add the product of the breadth and depth of each end; $\frac{1}{6}$ of this sum multiplied by the length, will give the exact solidity of any piece of squared timber tapering regularly.*

5. How many feet in a tree, whose ends are rectangles, the length and breadth of one being 14 and 12 inches, and their corresponding sides of the other 6 and 4 inches; also the length $30\frac{1}{2}$ feet?

14	12	$12 \times 14 = 168$
6	4	$6 \times 4 = 24$
—	—	$20 \times 16 = 320$
20	16	——
		512 square inches

$= \frac{32}{9}$ square feet.

Then $\frac{32}{9} \times 30\frac{1}{2} = 18\frac{2}{27}$ feet, the solidity.

6. How many solid inches in a mahogany plank, the length and breadth of one end being $81\frac{1}{2}$ and 55 inches, the length and breadth of the other end 41 and $29\frac{1}{2}$ inches, and the length of the plank $47\frac{1}{4}$ inches?

Ans. 126340·5937 cubic inches.

PROBLEM III.

Given the breadth of a rectangular plank in inches, to find how much in length will make a foot, or any other required quantity.

RULE. Divide 144, or the area to be cut off, by the breadth in inches, and the quotient will be the length in inches.

---

* This rule is correct, being that given for finding the solidity of the prismoid, or frustrum of a pyramid—which see.

Let B and b be the breadths of the two ends, D and d the depths, and L the length; $\frac{1}{6}(BD + (B+b) \times (D+d) + bd) \times L =$ the true solidity, as in the rule for the prismoid.

The Carpenters' rule is furnished with a scale which answers the purpose of this rule. It is called a table of board measure, and is in the following form:

0	0	0	0	5	0	$8\frac{1}{2}$	6	Inches.
12	6	4	3	2	2	1	1	Feet.
1	2	3	4	5	6	7	8	Breadth.

If the breadth be 1 inch, the length standing against it is 12 feet; if the breadth be 2 inches, the length standing against it is 6 feet; if the breadth be 5 inches, the length is 2 feet 5 inches, &c.

When the breadth goes beyond the limits of the table on the rule, it must be shut, and then you are to look for the breadth in the line of board measure, which runs along the rule from the table of board measure, and over against it on the opposite side, in the scale of inches, will be found the length required. For example, if the breadth be 9 inches, you will find the length a gainst it to be 16 inches; if the breadth be 11 inches, the length will be found to be a little above 13 inches.

1. If a board be 6 inches broad, what length of it will make a square foot? *Ans.* 2 feet.

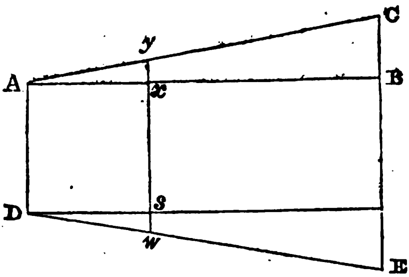
2. If a board be 8 inches broad, what length of it will make 4 square feet? *Ans.* 6 feet.

3. If a board be 16 inches broad, what length of it will make 7 square feet? *Ans.* $5\frac{1}{4}$ feet.

When the board is broader at one end than at the other, proceed as follows:

If it were required to cut off 60 square inches from the smaller end of a board, A D being 3 inches, C E 6 inches, and A B 20 inches.

By similar triangles, we have A B (20) : B C ($1\frac{1}{2}$) :: $Ax : xy = \frac{BC \times Ax}{AB} = \frac{1\frac{1}{2}\,Ax}{20} = \frac{3\,Ax}{40}$. But $Ax \times xy =$ the areas of the triangles Axy and DwS; therefore $\frac{3\,Ax}{40} \times Ax = \frac{3\,Ax^2}{40}$ is the sum of the areas of the trian-



gles $A\,x\,y$ and $D\,w\,S$. Now the rectangle $D\,S\,x\,A$, together with the adjacent triangles is equal to 60 square inches; therefore we have $\frac{3\,A\,x^2}{40} + 3\,A\,x = 60$ square inches, and $3\,A\,x^2 + 120\,A\,x = 2400$, and $A\,x^2 + 40\,A\,x = 800$; complete the square, and $A\,x^2 + 40\,A\,x + 400 = 1200$; extract the root, and $A\,x + 20 = \sqrt{1200} = 34{\cdot}64$; then $A\,x = 34{\cdot}64 - 20 = 14{\cdot}64$ is the length required.

PROBLEM IV.

To find how much in length will make a solid foot, or any other required quantity, of squared timber, of equal dimensions from end to end.

RULE. Divide 1728, the solid inches in a foot, or the solidity to be cut off, by the area of the end in inches, and the quotient will be the end in inches.

1. If a piece of timber be 10 inches square, how much in length will make a solid foot?

$10 \times 10 = 100$ the area of the end; then $1728 \div 100 = 17{\cdot}28$ *Ans.*

2. If a piece of timber be 20 inches broad, and 10 inches deep, how much of it will make a solid foot? *Ans.* $8\frac{1}{2}\frac{6}{5}$ ft.

3. If a piece of timber be 9 inches broad, and 6 inches deep, how much of it will make 3 solid feet? *Ans.* 8 ft.

On some Carpenters' rules, there is a table to answer the purpose of the last rule; it is called a Table of Timber, and is in the following form:

0	0	0	0	0	0	11	3	9	Inches.
144	36	16	9	5	4	2	2	1	Feet.
1	2	3	4	5	6	7	8	9	Side of square.

PROBLEM V.

To find the solidity of round or unsquared timber.

RULE I. Gird the piece of timber to be measured round the middle with a string, take ¼ part of the girth, and square it, and multiply this square by the length for the solidity.

BY THE SLIDING RULE.

As the length on C : 12 or 10 on D :: quarter girt, in 12ths or 10ths on D : content on C.

Note. When the tree is very irregular, divide it into several lengths, and find the solidity of each part separately; or add all the girts together, and divide the sum by the number of them.

1. Let the length of a piece of round timber be 9 feet 6 inches, and its mean quarter girt 42 inches; what is its content?

3·5 quarter girt.	3 : 6 quarter girt.
3·5	3 : 6
12·25	10 : 6
9·5 length.	1 : 9
116·375 content.	12 : 3
	9 : 6 length.
	110 : 3
	6 : 1 : 6
	116 : 4 : 6 content.

BY THE SLIDING RULE.

	C		D		D		C	
As	9·5	:	10	::	35	:	116⅓	content.
Or	9·5	:	12	:	42	:	116⅓	content.

RULE II. Multiply the area corresponding to the quarter girt in inches, by the length of the piece in feet, and the product will be the solidity.

Note. It may sometimes happen that the quarter girt exceeds the limits of the table, in this case, take half of it, and four times the content thus found will give the required content.

A TABLE FOR MEASURING TIMBER.

Quarter GIRT.	AREA.	*Quarter* GIRT.	AREA.	*Quarter* GIRT.	AREA.
Inches.	*Feet.*	*Inches.*	*Feet.*	*Inches.*	*Feet.*
6	·250	12	1·000	18	2·250
6¼	·272	12¼	1·042	18½	2·376
6½	·294	12½	1·085	19	2·506
6¾	·317	12¾	1·129	19½	2·640
7	·340	13	1·174	20	2·777
7¼	·364	13¼	1·219	20½	2·917
7½	·390	13½	1·265	21	3·062
7¾	·417	13¾	1·313	21½	3·209
8	·444	14	1·361	22	3·362
8¼	·472	14¼	1·410	22½	3·516
8½	·501	14½	1·460	23	3·673
8¾	·531	14¾	1·511	23½	3·835
9	·562	15	1·562	24	4·000
9¼	·594	15¼	1·615	24½	4·168
9½	·626	15½	1·668	25	4·340
9¾	·659	15¾	1·722	25½	4·516
10	·694	16	1·777	26	4·694
10¼	·730	16¼	1·833	26½	4·876
10½	·766	16½	1·890	27	5·062
10¾	·803	16¾	1·948	27½	5·252
11	·840	17	2·006	28	5·444
11¼	·878	17¼	2·066	28½	5·640
11½	·918	17½	2·126	29	5·840
11¾	·959	17¾	2·187	29½	6·044

2. If a piece of round timber be 10 feet long, and the quarter girt 12½ inches; required the solidity? *Ans.* 10·85.

To find the solid content by this Table, look for the quarter girt 12½, in the column marked, quarter girt, and in adjoining column marked, area, will be found 1·085, which multiplied by the length, 10 feet, will give 10·85 feet for the solid content.

3. A piece of round timber is 20 feet long, and the quarter girt 14¼; how many feet contained therein?

Ans. 28·2 feet.

4. How many solid feet are contained in a tree 40 feet long, its quarter girt being 9 inches *Ans.* 22·48 feet.

5. How many solid feet in a tree 32 feet long, its quarter girt being 8 inches? *Ans.* 14·208.

6. How many solid feet in a tree 8½ feet long, its quarter girt being 7½ inches? *Ans.* 3·315 feet.

7. Required the content of a tree, whose length is 40 feet, and quarter girt 27½ inches? *Ans.* 210·08 feet.

8. What is the content of a tree, whose length is 30 feet 6 inches, and quarter girt 27½ inches? *Ans.* 160·186 ft.

9. Required the content of a piece of timber, whose length is 25 feet 9 inches, and quarter girt 12¾ inches?

Ans. 29·071 feet.

10. What is the solid content of a piece of timber, whose length is 12 feet, and quarter girt 13½ inches?

Ans. 15·18 feet.

11. What is the solid content of a piece of timber, whose quarter girt is 14¾ inches, and length 38 feet?

Ans. 57·418 feet.

When the square of the quarter girt is multiplied by the length, the product gives a result nearly ¼ less than the true quantity in the tree. This rule, however, is invariably practiced by timber merchants, and is not likely to be abolished. When the tree is in the form of a cylinder, its content ought to be found by Prob. IV. Sec. V., which gives the content greater than that found by the last rule, nearly in the proportion of 14 to 11. Notwithstanding that the true content

is not found by means of the square of the quarter girt, yet some allowance ought to be made to the purchaser on account of the waste in squaring the wood so as to be fit for use. If the cylindrical tree be reckoned no more than what the inscribed square will amount to, the last rule, which is said to give too little, gives too much. When the tree is not perfectly circular, the quarter girt is always too great, and therefore the content, on that account, will be too great.

Doctor HUTTON recommends the following rule, which will give the content extremely near the truth:

RULE. Multiply the square of $\frac{1}{5}$ of the girt, or circumference by twice the length, and the product will be the content.

BY THE SLIDING RULE.

As double the length on C : 12 or 10 on D :: $\frac{1}{5}$ of the girt, in 12ths or 10ths on D : content on C.

12. Required the content of a tree, its length being 9 feet 6 inches, and its mean girt 14 feet?

$$14 \div 5 = 2{\cdot}8 = \overset{\text{ft.}}{2} : \overset{\text{in.}}{9} : \overset{\text{p.}}{7} = \tfrac{1}{5} \text{ of the girt; then}$$

	ft. in.	
2·8	9 : 6	2 : 9 : 7
2·8	2	2 : 9 : 7
7·84	19 : 0	5 : 7 : 2
19		2 : 1 : 2 : 3
		1 : 7 : 7 : 1
148·96 content.		7 : 9 : 11 : 10 : 1
		19
		148 : 9 : 8 : 11 : 7 content.

	C		D		D		C	
As	19	:	10	::	28	:	149	content.
Or	19	:	12	::	33·6	:	149	content.

Dr. GREGORY recommends the following rules given by *Mr. Andrews*:

Let L denote the length of the tree in feet and decimals, and G the mean girt, in inches.

RULE I. Making no allowance for bark.

$$\frac{L\,G^2}{2304} = \text{cubic feet, customary; and } \frac{L\,G^2}{1807} = \text{cubic feet, true content.}$$

RULE II. Allowing $\frac{1}{8}$ for bark.

$$\frac{L\,G^2}{3009} = \text{cubic feet, customary; } \frac{L\,G^2}{2360} = \text{cubic feet, true content.}$$

RULE III. Allowing $\frac{1}{10}$ for bark.

$$\frac{L\,G^2}{2845} = \text{cubic feet, customary; } \frac{L\,G^2}{2231} = \text{cubic feet, true content.}$$

RULE IV. Allowing $\frac{1}{12}$ for bark.

$$\frac{L\,G^2}{2742} = \text{cubic feet, customary; } \frac{L\,G^2}{2150} = \text{cubic feet, true content.}$$

What is the solid content of a tree, whose circumference, or girt, is 60 inches, and length 40 feet?

BY RULE I.

$$\frac{40 \times 60^2}{2304} = 62\tfrac{1}{2} \text{ cubic feet, customary.}$$

$$\frac{40 \times 60^2}{1807} = 79\tfrac{3}{4} \text{ cubic feet, true content.}$$

BY RULE II.

$$\frac{40 \times 60^2}{3009} = 47{\cdot}85 \text{ cubic feet, customary.}$$

$$\frac{40 \times 60^2}{2360} = 61 \text{ cubic feet, true content.}$$

By Rule III.

$$\frac{40 \times 60^2}{2845} = 50 \cdot 61 \text{ cubic feet, customary.}$$

$$\frac{40 \times 60^2}{2231} = 64 \cdot 54 \text{ cubic feet, true content.}$$

By Rule IV.

$$\frac{40 \times 60^2}{2742} = 52 \cdot 47 \text{ cubic feet, customary.}$$

$$\frac{40 \times 60^2}{2150} = 66 \cdot 97 \text{ cubic feet, true content.}$$

When the two ends are very unequal, calculate its content by the rule given for finding the solidity of the frustrum of a cone, and deduct the usual allowance from the result.

When it is required to find the accurate content of an irregular body, not reducible to any figure of which we have already treated, provide a cylindrical or prismatic vessel, capable of containing the solid to be computed; put the solid into the vessel, and pour in water to cover it, marking the height to which the water reaches. Then take out the solid, and observe how much the water has descended in consequence of its removal; calculate the capacity of the part of the vessel thus left dry, and it will be evidently equal to the solidity of the body, whose content is required.

ARTIFICERS' WORK.

Artificers compute their works by several different measures:

Glazing and masonry by the foot.

Plastering, painting, paving, &c. by the yard of 9 square feet.

Partitioning, roofing, tiling, flooring, &c. by the square of 100 square feet.

Brick work is computed, either by the yard of 9 square feet, or by the perch, or square rood, containing $272\frac{1}{4}$ square feet, or $30\frac{1}{4}$ square yards; $272\frac{1}{4}$ and $30\frac{1}{4}$ being the squares *of* $16\frac{1}{2}$ *feet* and $5\frac{1}{2}$ yards respectively.

CARPENTERS' AND JOINERS' WORK.

1. Of Flooring.

To measure joists, multiply the breadth, depth, and length together, for the content.*

If a floor be 50 feet 4 inches long, and 22 feet 6 inches broad; how many squares of flooring in that room?

$$\begin{array}{r} 50\cdot3 \\ 22\cdot5 \\ \hline 2515 \\ 1006 \\ 1006 \\ \hline 100)1131\cdot75 \\ \hline 11\cdot3175 \text{ squares.} \end{array} \qquad \begin{array}{r} 50 : 4 \\ 22 : 6 \\ \hline 1107 : 4 \\ 25 : 2 \\ \hline 100)11,32 : 6 \\ \hline 11.32 : \end{array}$$

Ans. 11 squares 32 feet.

2. If a floor be 51 feet 6 inches long, and 40 feet 9 inches broad; how many squares contained in that floor?

Ans. 20·986 squares.

3. If a floor be 36 feet 3 inches long, and 16 feet 6 inches broad; how many squares are contained in that floor?

Ans. 5 squares 98$\frac{1}{8}$ feet.

4. If a floor be 86 feet 11 inches long, and 21 feet 2 inches broad; how many squares are contained in it?

Ans. 18·3972.

5. In a naket floor the girder is 1 foot 2 inches deep, 1 foot broad, and 22 feet long; there are 9 bridgings, whose scantling (viz. breadths and depths,) are 3 inches, by 6 inches, and 22 feet long; 9 binding joists, whose lengths

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* Joists receive various names from their position; such as girders, binding-joists, trimming-joists, common-joists, ceiling-joists, &c. When girders and joists of flooring are designed to bear considerable weight, they should be let into the wall at each end about $\frac{2}{3}$ of the thickness of the wall.

are 10 feet, and scantlings 8 inches by 4 inches; the ceiling joists are 25 in number, each 7 feet long, and their scantlings 4 inches by 3 inches; what is the solidity of the whole? *Ans.* 85 feet.

6. What would the flooring of a house three stories high come to, at £5 per square; the house measures 30 feet long, and 20 broad; there are seven fire places, two of which measure, each 6 feet by 4 feet, two others, each 6 feet by 5 feet 6 inches; two, each of 5 feet 6 inches by 4 feet; and the seventh 5 feet by 4; the well-hole for the stairs is 10 feet by 8? *Ans.* £39 8*s*.

Of Partitioning.

Partitions are measured by squares of 100 feet, as flooring; their dimensions are taken by measuring from wall to wall, and from floor to floor; then multiply the length and height for the content in feet, which bring to squares by dividing 100, as in flooring. When doors and windows are not included by agreement, deductions must be made for their amount.*

1. A partition measures 173 feet 10 inches in length, and 10 feet 7 inches in height; required the number of squares in it? *Ans.* 18·3972 squares.

2. A partition between two rooms measures 80 feet in length, and 50 feet 6 inches in height; how many squares in it? *Ans.* $40\frac{2}{5}$ squares.

3. If a partition measure 10 feet 6 inches in length, and 10 feet 9 inches in height; how many squares in it? *Ans.* 1 square $12\frac{7}{8}$ feet.

4. What is the number of squares in a partition, whose length is 50 feet 6 inches, and height 12 feet 9 inches? *Ans.* 6 squares 43 feet $10\frac{1}{2}$ inches.

In roofing, the length of the rafters is equal to the length of a string stretched from the ridge down the rafter till it meets the top of the wall.

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* *The best and strongest partitions are those made with framed-work. The king-posts are measured as roofing, the rest as flooring.*

To find the content, multiply this length by the breadth and depth of the rafters, and the result will be the content of one rafter; and that multiplied by the number of them will give the content of all the rafters.*

1. If a house within the walls be 42 feet 6 inches long, and 20 feet 3 inches broad; how many squares of roofing in that house?

42·5	42 : 6
20·25	20 : 3
2125	840
850	$6\frac{1}{2}$ 10 : 1
8500	$3\frac{1}{4}$ 10 : 7
860·625 flat.	860 : 8 flat.
430·3125	430 : 4
100)1290·9375	100)1291
12·91 squares.	12 : 91 squares.

2. What cost the roofing of a house at 11*s.* per square; the length within the walls being 50 feet 9 inches, and the breadth 30 feet; the roof being of a true pitch?

Ans. £12 11*s.* $2\frac{11}{50}$.

3. What number of squares are contained in a house, whose length within the walls is 40 feet, and breadth 18 feet; the roof being common pitch?

Ans. 10 squares and 80 feet.

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* Workmen generally take the flat and half the flat of any house, taken within the walls, to be the measure of the roof of the same house. This, however, is only when the roof is of a true pitch. The usual pitches are the common, or true pitches, in which the rafters are $\frac{3}{4}$ of the breadth of the builing; the Gothic pitch, is when the length of the principal rafters is equal to the breadth of the building; the pediment pitch, is when the perpendicular height is $\frac{2}{9}$ of the breadth.

When the covering of the building is to be plain tiles, or slates, the roof is generally of a true or common pitch; the Gothic pitch is used, when the covering is of pantiles; the pediment pitch is used, when the roof is to be covered with lead.

4. How many squares in the roof of a building, the length of the house being 60 feet, and the length of the rafter 14 feet 6 inches? *Ans.* 17 squares and 40 feet.

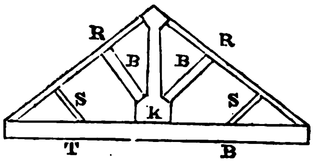
5. How many squares in a building, whose length is 50 feet, and the length of the rafter 15 feet? *Ans.* 15 squares.

6. How many squares in the roof of a building, whose length is 37 feet, the length of the rafter being 13 feet? *Ans.* 9 squares and 62 feet.

7. How many squares in the roof of a building, whose length is 70 feet 6 inches, the length of the rafter being 14 feet 6 inches? *Ans.* 20 squares and $44\frac{1}{2}$ feet.

8. How many squares in the roof of a building, whose length is 50 feet, and the length of a string reaching across the ridge from eave to eave being 30 feet? *Ans.* 15 squares.

Note. All the timbers employed in roofing are measured like those used in flooring, except where there is a necessity of cutting out parallel pieces equal to, or exceeding $2\frac{1}{2}$ inches broad and 2 feet long. In this case the amount of the pieces so cut out must be deducted from the content of the whole piece found from its greatest scantlings. When the pieces cut out do not amount to the above dimensions, they are considered as useless, and therefore no deduction is to be made for them.*



10. Let the tie-beam T D be 36 feet long, 9 inches broad, and 1 foot 2 inches thick; the king post K 11 feet 6 inches high, 1 foot broad at the bottom, and 5 inches

* In the above figure, K is called the king-post, and in measuring the pieces cut out of it, the shortest length is to be taken. T, B, is called the tie-beam, which prevents the rafters R, R, from pressing out the wall. The braces B, B, serve to strengthen the rafters; the struts S, S, serve for a similar purpose. Besides strengthening the rafters, the braces and struts serve to bind the roof together. When *head room is* required, the rafters are braced simply by R, R.

thick; out of this post are sawn two equal pieces from the sides, each 7 feet long and 3 inches broad. The braces B, B, are 7 feet 6 inches long, and 5 inches by 5 inches square; the rafters R, R are 19 feet long, 5 inches broad, and 10 inches deep; the struts S, S, are 3 feet 6 inches long, 4 inches broad, and 5 inches deep; what is the measurement for workmanship, and also for materials?

ft.	in.	p.	
31 :	6 :	0	solidity of the tie-beam T D.
4 :	9 :	6	solidity of the king-post K.
2 :	7 :	3	solidity of the braces B, B.
13 :	2 :	4	solidity of the rafters R, R.
	11 :	8	solidity of the struts S, S.
53 :	0 :	9	solidity for workmanship.
1 :	5 :	6	solidity cut from the king-post.
51 :	7 :	3	solidity for materials.

Of Wainscotting.

Wainscotting is measured by the yard square, which is 9 square feet.

In taking the dimensions, the string is made to ply close over the cornice, swelling pannels, moulding, &c. The height of the room from the floor to the ceiling being thus taken, is one dimension, and the compass of the room taken all round the floor is the second dimension.

Doors, windows, shutters, &c. where both their sides are planed, are considered as work and half; therefore in measuring the room, they need not be deducted; but the superficial content of the whole room found as if there were no door, window, &c.; then the contents of the doors and windows must be found, and half thereof added to the content of the whole room.

When there are no shutters, the content of the windows must be deducted; chimneys, windows-seats, cheek-boards, sopheta-boards, linings, &c. must be measured by themselves.

Windows are sometimes valued at so much per window, and sometimes by the superficial foot. The dimensions of a

window are taken in feet and inches, from the under side of the sill to the upper side of the top-rail; and from the outside to outside of the jambs.

When the doors are panelled on both sides, take double the measure for the workmanship.

For the surrounding architrave, girt round it and inside the jambs, for one dimension, and add the length of the jambs to the length of the cap-piece, (taking the breadth of the opening for the length,) for the other dimension.

Weather-boarding is measured by the yard square, and sometimes by the square.

Frame-doors are measured by the foot, or sometimes by the yard square.

Stair-cases are measured by the foot superficial. The dimensions are taken with a string passing over the riser and tread for one dimension, and the length of the step for the other. By the length of the step is meant the length of the front and the returns at the two ends.

For the ballustrade, take the whole length of the upper part of the hand-rail, and girt it over its end till it meet the top of the newel post, for one dimension; and twice the length of the baluster upon the landing, with the girt of the hand-rail, for the other dimension.

Frame-doors are measured by the foot, or sometimes by the yard square.

Madillian cornices, coves, &c. are generally measured by the foot superficial.

Beads, stops, astragals, copings, fillets, boxings to windows, skirting boards, and water-trunks, are paid for by lineal measure.

Frontispieces are measured by the foot superficial, and the architrave, frieze, and cornice are measured separately.*

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* Baluster is a small column or pillar, used for balustrades.

Balustrade is a row of balusters, joined by a rail; serving for a rest to the arms, or as an inclosure to balconies, staircases, altars, &c.

Cornice is the third and uppermost part of the entablature of a column, or the uppermost ornament of any wainscotting, &c.

Bead is a round moulding carved like beads in necklaces. There is also a kind of plain bead, often set on the edge of each fascia of

To find the contents of the foregoing work, multiply the two corresponding dimensions together for the superficial content.

1. A room, or wainscot, being girt downwards over the mouldings, measures 12 ft. 6 in., and 130 ft. 9 in. in compass; how many yards does that room contain.

ft.	in.	
130 :	9	
12 :	6	
1560		
65 :	4 :	6
6 :	0 :	0
3 :	0 :	0
9)1634 :	4 :	6
181 :	5 *Ans.*	

130·75
12·5
65375
26150
13075
9)1634·375 ft.
181 yards, 5 ft.

2. If the wainscot of a room be 15 ft. 6 in. high, and the compass of the room 142 ft. 6 in.; how many yards are contained in it? *Ans.* $245\frac{5}{12}$ yards.

3. If the window shutters about a room be 60 ft. 6 in. broad, and 6 ft. 4 in. high; how many yards are contained therein, at work and a-half? *Ans.* $63\frac{31}{36}$ yards.

4. A rectangular room measures 129 feet 6 inches round, and is to be wainscotted at 3s. 6d. per square yard; after

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an architrave, on the upper edge of skirting boards, on the lining board of a door-case, &c.

Architrave, is that part of a column that bears immediately on the capital. It is supposed to represent the principal beam in timber buildings, in which it is sometimes called the master-piece, or reason-piece. In chimneys it is called the mantel-piece. Architrave doors, are those which have an architrave on the jambs and over the doors. Architrave windows of timber are usually raised out of the solid timber, and sometimes the mouldings are struck and laid on.

Astragal is a small round moulding, encompassing the top of the shaft of a column, like a ring or bracelet. The shaft terminates at the top with an astragal, and at bottom with a fillet, which in this place is called apophysis.

due allowance for girt of cornice, &c., it is 16 feet 3 inches high; the door is 7 feet by 3 feet 9 inches; the window shutters, two pair, are 7 feet 3 inches by 4 feet 6 inches; the cheek-boards round them come 15 inches below the shutters, and are 14 inches in breadth; the lining-boards round the doorway are 16 inches broad; the door and window shutters being worked on both sides, are reckoned as work and half, and paid for accordingly; the chimney 3 feet 9 inches by 3 feet, not being enclosed, is to be deducted from the superficial content of the room. The estimate of the charge is required. *Ans.* £43 4*s.* 6¾*d.*

The height of a room, taking in the cornice and mouldings, is 12 feet 6 inches, and the whole compass 83 feet 8 inches; the three window shutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the door and shutter, being worked on both sides, are reckoned work and a half. Required the estimate, at 6*s.* per square yard. *Ans.* £36 12*s.* 2½*d.*

OF BRICKLAYERS' WORK.

Of Tiling, or Slating.

Tiling and slating are measured by the square of 100 feet. There is no material difference between the method employed for finding the estimate of roofing and tiling; bricklayers sometimes require double measure for hips and valleys.

When gutters are allowed double measure, the usual mode is to measure the length along the ridge-tile, and add it to the contents of the roof: this makes an allowance of one foot in breadth, along the hips or valleys. Double measure is usually allowed for the eaves, so much as the projector is over the plate, which is generally 18 or 20 inches.

When sky-lights and chimney-shafts are not large, no

allowance is to be made for them; but when they are large, their amount is to be deducted.

1. There is a roof covered with tiles, whose depth on both sides (with the usual allowance at the eaves) is 30 feet 6 inches, and the length 42 feet; how many squares of tiling are contained therein?

ft. in.	
30 : 6	30·5
42	42
1260	610
21	1220
100)12,81	100)12,810
12 : 81	12 squares, 81 feet.

2. There is a roof covered with tiles, whose depth on both sides (with the usual allowance at the eaves) is 40 feet 9 inches, and the length 47 feet 6 inches; required the number of squares contained therein. *Ans.* 19 squares $35\frac{5}{8}$ feet.

3. What will the slating of a house cost at £1 5*s.* 6*d.* per square; the length being 43 feet 10 inches, and the breadth 27 feet 5 inches, on the flat; the eaves projecting 16 inches on each side—true pitch. *Ans.* £24 9*s.* $5\frac{1}{2}$*d.*

4. What is the contents of a slated roof, the length being 45 feet 9 inches, and the whole girt 34 feet 3 inches.

Ans. 174·104 yards.

Of Walling.

Brick-work is estimated at the rate of a brick and a half thick; so that if a wall be more or less than this standard thickness, it must be reduced to it: thus, multiply the superficial contents of the wall by the number of half bricks in the thickness, and divide the product by 3.

The superficial contents is found by multiplying the length by the height. Bricklayers estimate their work by the rod of $16\frac{1}{2}$ feet, or $272\frac{1}{4}$ square feet. Sometimes 18 feet are allowed to the rod; that is, 324 square feet; sometimes the work is measured by the rod of 21 feet long, and 3 feet high: that is, 63 square feet: in this case, sometimes no

regard is paid to the thickness of the wall; but the price is regulated according to the thickness.

When a piece of brick-work is to be measured, the first thing to be done, is to ascertain which of the above measures is to be employed; then, having multiplied the length and breadth together (the dimensions being feet) the product is to be divided by the proper divisor, namely 272·25; 324; or 63; according to the measure of the rod, and the quotient will be the measure in square rods of that measure.

To measure any arched way, arched window, or door, &c., the height of the window, or door, from the crown or middle of the arch, to the bottom or sill, is to be taken; and likewise from the bottom or sill to the spring of the arch; that is, where the arch begins to turn. Then to the latter height, add twice the former, and multiply the sum by the breadth of the window, door, &c., and one-third of the product will be the area sufficiently near the truth for practice.

1. If a wall be 72 feet 6 inches long, and 19 feet 3 inches high, and 5 bricks and a half thick; how many rods of brick-work are contained therein, when reduced to the standard?

```
        ft.  in.
        72 : 6
        19 : 3
       --------
       648
       72
        18 : 1 : 6
         9 : 6 : 0
      -------------
      1395 : 7 : 6
                11
    ---------------
   3)15351 : 10 : 6
   -------
  272)5117(18 rods.
      ----
      2397
     -----
    68)221(3 quarters.
       ---
        17 feet.
```

Note. That 68·06 is the fourth part of 272·26, and 68 is one fourth of 272.

In reducing feet into rods, it is usual to divide 272, rejecting the decimal ·25. By this method, the answer found above is about $4\frac{1}{2}$ feet too much.

2. How many rods of standard brick-work are in a wall whose length is 57 feet 3 inches, and height 24 feet 6 inches; the wall being $2\frac{1}{2}$ bricks thick. *Ans.* 8·5866 rods.

3. The end wall of a house is 28 feet 10 inches long, and 55 feet 8 inches high to the eaves, 20 feet high is $2\frac{1}{2}$ bricks thick, the other 20 feet high is 2 bricks thick, and the remaining 15 feet 3 inches is $1\frac{1}{2}$ brick thick, above which is a triangular gable one brick thick, which rises 42 courses of bricks, of which every 4 courses make a foot. What is the whole content in standard measure? *Ans.* 253·62 yards.

Of Chimneys.

When a chimney stands by itself, without any party-wall being adjoined, take the girt in the middle for the length, and the height of the story for the breadth; the thickness is to be the same as the depth of the jambs; if the chimney be built upright from the mantel-piece to the ceiling, no deduction is to be made for the vacancy between the floor (or hearth) and mantel-tree, on account of the gatherings of the breast and wings, to make room for the hearth in the next storey.

When the chimney-back forms a party-wall, and is measured by itself, then the depth of the two jambs is to be measured, and the length of the breast, for a length, and the height of the storey for the breadth; the thickness is the same as the depth of the jambs. That part of the chimney which appears above the roof, called the chimney-shaft, is measured by girding it round the middle for the length, and the height is taken for the breadth.

In consideration of plastering and scaffolding, the thickness is generally reckoned half a brick more than it really is; and in some places double measure is allowed, on account of extra trouble.

1. Let the dimensions of a chimney, having a double funnel towards the top, and a double shaft, be as follows:

viz., In the parlour, the breast and two jambs measure 18 feet 9 inches, and the height of the room is 12 feet 6 inches; in the first floor, the breast and two jambs girt 14 feet 6 inches, and the height 9 feet; in the second floor, the breast and the jambs girt 10 feet 3 inches, and the height is 7 feet; above the roof, the compass of the shaft is 13 feet 9 inches, and its height 6 feet 6 inches; lastly, the length of the middle partition, which parts the funnel, is 12 feet, and its thickness 1 foot 3 inches; how many rods of brick-work, standard measure, are contained in the chimney, double measure being allowed, and thickness $1\frac{1}{2}$ brick?

```
        ft.  in.                    ft.  in. p.
1       18 : 9                       1 : 3 : 0
        12 : 6                      12
        ------                      ---------
       225 : 0                      15 : 0 partition.
         9 : 4 : 6                 234 : 4 : 6 parlour.
       ----------                  130 : 6 : 0 first floor.
       234 : 4 : 6                  71 : 9 : 0 second floor.
                                    89 : 4 : 6 shaft.
        ft.  in.                   -----------
2.      14 : 6                     541 : 0 : 0 sum.
         9                                   2
        ------                 --------------
       130 : 6              272)1082 : 0 : 0 double.
                                ------------
        ft.                    68)266(3 rods 3 quarters.
        10 : 3                    ---
         7                        62 feet.
        ------
        71 : 9

        ft.  in.
        13 : 9
         6 : 6
        ------
        82 : 6
         6 : 10 : 6
        -----------
        89 :  4 : 6
```

Ans. 3 rods, 3 quarters, and 62 feet.

MASONS' WORK.

To masonry belong all sorts of stone-work. The work is sometimes measured by the foot solid, sometimes by the foot in length, and sometimes by the foot superficial. Masons, in taking dimensions, girt all their mouldings, similar to the practice of joiners.

Walls, columns, blocks of stone or marble, &c., are measured by the solid foot, and pavements, slabs, chimney-pieces, &c., by the square foot.

In estimating for the workmanship, square measure is generally used, but for the materials, solid measure.

In the solid measure, the length, breadth, and thickness are multiplied together.

In the superficial measure, there must be taken the length and breadth of every part of the projection, which is seen without the general upright face of the building.

1. If a wall be 82 feet 9 inches long, 20 feet 3 inches high, and 2 feet 3 inches thick; how many solid feet are contained in that wall?

	ft. in.	ft.
	82 : 9	82·75
	20 : 3	20·25
	1640	41375
$3 = \frac{1}{4}$	20 : 8 : 3	16550
$6 = \frac{1}{2}$	10 : 0 : 0	165500
$3 = \frac{1}{2}$	5 : 0 : 0	
		1675·6875
	1675 : 8 : 3	2·25
	2 : 3	
		83784375
	3351 : 4 : 6	33513750
$3 = \frac{1}{4}$	418 : 11 : 0¼	33513750
	3770 : 3 : 6¼	3770·296875 *Ans.*

2. If a wall be 120 feet 4 inches long, and 30 feet 8 inches high; how many superficial feet are contained therein?

Ans. $3690\frac{2}{9}$ feet.

3. If a wall be 112 feet 3 inches long, and 16 feet 6 inches high; how many superficial rods of 63 square feet are contained therein? *Ans.* 29 rods 25 feet.

4. What is the value of a marble slab at 8*s.* per foot, the length being 5 feet 7 inches, and breadth 1 foot 10 inches? *Ans.* £4 1*s.* 10½*d.*

PLASTERERS' WORK.

Plasterers' work is of two kinds, viz. ceiling, which is plastering upon laths; and rendering, which is plastering upon walls. These are measured separately.

The content is sometimes estimated by the foot, sometimes by the yard, and sometimes by the square of 100 feet. Enriched mouldings are calculated by the running foot or yard.

Deductions are made for chimneys, doors, windows, &c.

In plastering timber partitions, where several of the large braces and other large timbers project from the plastering, a fifth is usually deducted.

Whitening and colouring are measured in the same manner as plastering. In timbered partitions, one-fourth, or one-fifth of the whole area is usually added, to compensate for the trouble of colouring the sides of the quarters and braces.

In arches, the girt round them is multiplied by the length for the superficial content.

1. If a ceiling be 40 feet 3 inches long, and 16 feet 9 inches broad; how many square yards contained therein?

40 : 3	40·25
16 : 9	16·75
640	20125
6 = ½ — 20 : 1 : 6	28175
3 = ½ — 10 : 0 : 9	24150
3 = ¼ — 4 : 0 : 0	4025
9)674 : 2 : 3	9)674·1875
Ans. 74 yards 8 feet.	*Ans.* 74·9097 yards.

2. The length of a room is 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 2 inches to the under side of the cornice, which projects 5 inches from the wall, on the upper part next the ceiling; required the quantity of rendering and plastering, there being no deduction but for one door, which is 7 feet by 4?

Ans. 53 yards 5 feet of rendering, 18 yards 5 feet of ceiling.

3. The circular vaulted roof of a church measures 105 feet 6 inches in the arch, and 275 feet 5 inches in length; what will the plastering come to at 1*s.* per yard?

Ans. £161 8*s.* 5¾d.

4. The length of a room is 18 feet 6 inches; what do the ceiling and rendering come to, the former at 8*d*, and the latter at 3*d.* per yard; allowing for the door, which is 7 feet by 3 feet 8 inches, and a fire-place of 5 feet square?

Ans. £1 13*s.* 3*d.*

PLUMBERS' WORK.

Plumbers' work is rated by the pound or hundred weight of 112lbs. sheet lead, used in roofing, guttering, &c., weight from 6 to 12 pounds per square foot, according to the thickness; and leaden pipes vary in weight per yard, according to the diameter of its bore in inches.

The following table shows the weight of a square foot of sheet lead, according to its thickness; and the common weight of a yard of leaden pipe according to the diameter of its bore.

Thickness of lead.	Pounds to a square foot.	Bore of leaden pipes	Pounds per yard.
Inch.			
$\frac{1}{10}$	5·899	0¾	10
$\frac{1}{9}$	6·554	1	12
$\frac{1}{8}$	7·373	1¼	16
$\frac{1}{7}$	8·427	1½	18
$\frac{1}{6}$	9·831	1¾	21
$\frac{1}{5}$	11·797	2	24

1. A piece of sheet lead measures 20 feet 6 inches in length, and 7 feet 9 inches in breadth; what is its weight at $8\frac{1}{4}$ lb to the square foot?

```
 ft.  in.
 20 : 6
  7 : 9
 ----------
143 : 6
 15 : 4 : 6
 ----------
158 : 10 : 6
```

```
           ft.
          20·5
           7·75
          ------
          1025
         1435
        1435
       ---------
       158·875
            8¼
      ----------
      1271·000
        39·719  ct.  qr.  lb.
   112)1310·719(11 : 2 : 22¾ nearly.
       112
       ---
        190
        122
       ----
     28)78(2
        56
        --
        22
```

2. What weight of lead $\frac{1}{10}$ of an inch thick will cover a flat, 15 feet 6 inches long, and 10 feet 3 inches broad, the lead weighing 6lb. to the square foot?

Ans. 8cwt. 2qrs. $1\frac{1}{4}$lb.

3. What will be the expense of covering and guttering a roof with lead, at 18*s.* per cwt. weight; the length of the roof being 43 feet, and the girt over it 32 feet; the guttering being 57 feet in length and 2 feet in breadth, allowing a square foot of lead to weigh $8\frac{3}{4}$ lb?

Ans. £104 15*s.* $3\frac{3}{4}$*d.*

4. What will be the expense of 130 yards of leaden pipe of an inch and half bore, at 4*d.* per lb, admitting each yard *to weigh* 18lb? *Ans.* £39.

PAINTERS' WORK.

Painters' work is computed in square yards. Every part is measured where the colour lies, and the measuring line is forced into all the mouldings and corners. Double measure is allowed for carved mouldings, &c.

Windows are done at so much a-piece. Sash-frames at a certain price per dozen; sky-lights, window-bars, casements, &c. are charged at a certain price per piece.

To measure ballustrades, take the length of the hand-rail for one dimension, and twice the height of the balluster upon the landing, added to the girt of the hand-rail, for the other dimension.

No general rule can be given for measuring trellis-work; but, however, double the area of one side is often taken for the measure of both sides.

1. If a room be painted, whose height (being girt over the moulding,) is 16 feet 4 inches, and the compass of the room 120 feet 9 inches; how many yards of painting in it?

	ft. in.	ft.
	120 : 9	120·75
	16 : 4	16·3
	1920	36225
$4 = \frac{1}{3}$ —	40 : 3	72450
$6 = \frac{1}{2}$	8 : 0	12075
$3 = \frac{1}{2}$	4 : 0	9)1968·225
	9)1972 : 3	*Ans.* 218·691 yards.

Ans. 219 yards 1 foot.

2. A gentleman had a room to be painted, its length being 24 feet 6 inches, breadth 16 feet 3 inches, and height 12 feet 9 inches; also the size of the door 7 feet by 3 feet 6 inches, and the size of the window-shutters to each of the windows, there being two, is 7 feet 9 inches by 3 feet 6 nches; but the breaks of the windows themselves are 8 feet 6 inches high, and 1 foot 3 inches deep; what will be the

expense of giving it three coats, at 2*d.* per yard each; the size of the fire-place to be deducted, being 5 feet by 5 feet 6 inches? *Ans.* £3 3*s.* 10½*d.*

3. The length of a room is 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two window shutters each 6 feet by 3 feet 2 inches? *Ans.* $73\frac{2}{27}$ yards.

GLAZIERS' WORK.

Glaziers take their dimensions either in feet, inches, and parts; or feet, tenths, and hundredths. They compute their work in square feet.

Windows are sometimes measured by taking the dimensions of one pane, and multiplying its superficies by the number of panes. But generally they take the length and breadth of the whole frame for the glazing. Circular windows are measured as if they were square, taking for their dimensions, their greatest length and breadth.

1. If a pane of glass be 3 feet 6 inches and 9 parts long, and 1 foot 3 inches and 3 parts broad; how many feet of glass in that pane?

3 : 6 : 9	3.56
1 : 3 : 3	1·277
3 : 6 : 9	2492
10 : 8 : 3	2492
10 : 8 : 3	712
	356
Ans. 4 : 6 : 3 : 11 : 3	
	Ans. 4·54612 feet.

2. If there be 10 panes of glass, each 4 feet 8 inches 9 parts long, and 1 foot 4 inches and 3 parts broad; how many feet of glass are contained in the 10 panes? *Ans.* 60·403.

3. There are 20 panes of glass, each 3 feet 6 inches 9

parts long, and 1 foot 3 inches and 3 parts broad; how many feet of glass are in the 20 panes? *Ans.* 90·9224 ft.

4. If a window be 7 feet 6 inches high, and 3 feet 4 inches broad; how many square feet of glass contained therein? *Ans.* 25.

5. How many feet in an elliptical fan-light of 14 feet 6 inches in length, and 4 feet 9 inches in breadth?
Ans. 68 feet 10 inches.

6. What will the glazing of a triangular sky-light come to at 20*d.*; the base being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches? *Ans.* £3 10*s.* 3½.

PAVERS' WORK.

Pavers' work is computed by the square yard; and the content is found by multiplying the length by the breadth.

1. What will be paid for paving a foot-path, at 4*s.* the yard, the length being 40 feet 6 inches, and the breadth 7 feet 3 inches?

ft. in.	ft.
40 : 6	40·5
7 : 3	7·25
283 : 6	2025
10 : 1 : 6	810
	2835
Ans. 293 : 7 : 6	
	Ans. 293·625 feet.

2. What will be the expense of paving a rectangular courtyard, whose length is 62 feet 7 inches, and breadth 44 feet 5 inches; and in which there is a foot-path, whose whole length is 62 feet 7 inches, and breadth 5 feet 6 inches, this at 3*s.* per yard, and the rest at 2*s.* 6*d.* per yard?
Ans. £39 11*s.* 3¼*d.*

3. What is the expense fo paving a court, at 3*s.* 2*d.* per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches? *Ans.* £7 4*s.* 5¼*d*

4. What will the paving of a walk round a circular bowling-green come to, at 2*s.* 4*d.* per yard, the diameter of the bowling-green being 40 feet, and the breadth of the walk 5 feet? *Ans.* £9 3*s.* $3\frac{3}{25}$*d.*

5. How many yards of paving in an elliptical walk 4 feet broad, the longer diameter being 60 feet, and shorter 50?
Ans. 82·3797 yards.

VAULTED AND ARCHED ROOFS.

Arched roofs are either domes, vaults, saloons, or groins.

Domes are formed of arches springing from a circular, or polygonal base, and meeting in a point directly over the centre of that base.

Saloons are made by arches connecting the side walls of a building to a flat roof, or ceiling.

Groins are made by the intersection of vaulted roofs with each other.

Vaulted roofs are sometimes circular, sometimes elliptical, and sometimes Gothic.

Circular roofs are those of which the arch is a part of the circumference of a circle.

Elliptical roofs are those of which the arch is a part of the circumference of an ellipsis.

Gothic roofs are made by the meeting of two equal circular arches, axactly above the span of the arch.

Groins are generally measured like a parallelopipedon, and the content is found by multiplying the length and breadth of the base by the height.

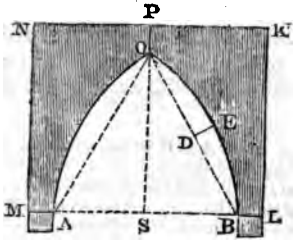
Sometimes one-tenth is deducted from the solidity thus found, and the remainder is reckoned as the solidity of the vacuity.

PROBLEM I.

To find the solidity of a circular, elliptical, or Gothic vaulted roof.

RULE. Find the area of one end, by one of the foregoing rules, and multiply the area of the end by the length of the roof, or vault, and the product will be the content.

Note. When the arch is a segment of a circle, the area is found by Prob. XXVIII. Sec. II. When the arch is a segment of an ellipsis; multiply the span by the height, and that product by ·7854 for the area of the end. When it is a Gothic arch, find the area of an isosceles triangle, whose base is equal to the span of the arch, and its sides equal to the two chords of the circular segment of the arch; then add the areas of the two segments to the area of the triangle, and the sum will give the area of the end.



1. What is the content of a concavity of a semi-circular vaulted roof, the span being 30 feet, and the length of the vault 150 feet?

$30 \times 30 = 900$; then $900 \times \cdot7854 = 706\cdot86$, hence $706\cdot86 \div 2 = 353\cdot43$ the area of the end; then $353\cdot43 \times 150 = 5301\cdot45$ the content.

2. What is the solid content of the vacuity A O E B of a Gothic vault, whose span A B is 60 feet, the chord B O, or A O, of each arch 60 feet; the distance of each arch from the middle of the chords as D E = 12 feet, and the length of the vault 40 feet?

In this example, the triangle A B O is equi-lateral, and its area is $\frac{1}{4}$ A B$^2\sqrt{3} = 900\sqrt{3} = 1557$. Again, $\frac{2}{3}$ (B O $\times$

$D\,E) + \frac{D\,E^3}{2\,B\,O} = \frac{2}{3}(60 \times 12) + \frac{12^3}{60 \times 2} = 494\frac{2}{5} =$ area of segment O E B, and $494\frac{2}{5} \times 2 = 988\frac{4}{5}$ the areas of the two segments O E B and O H A; then $(1557 + 988\frac{4}{5}) \times 40 = 101832$ the solidity required.

Let M N K L represent a perpendicular section of a vaulted roof, (Gothic). The span A B is 60 feet, the thickness of the wall M A, or B L, at the spring of the arch = 4 feet, the thickness O P at the crown of the arch = 3, and the length of the roof = 40 feet, the chord A O or O B = 60 feet, and the versed sine D E = 12 feet; required the solidity of the materials of the arch?

First, $\sqrt{(A\,O^2 - A\,C^2)} = \sqrt{(60^2 - 30^2)} = 51{\cdot}96 =$ S O the height of the vacuity of the arch, and $S\,O + O\,P = 51{\cdot}96 + 3 = 54{\cdot}96 = S\,P$; again, $A\,B + M\,A + B\,L = 60 + 4 + 4 = 68 = M\,L$, and $M\,L \times S\,P =$ the area of the rectangle M N K L; hence, $M\,L \times S\,P \times 40 - 101832$, (the solidity of the vacuity A O B by the last Problem,) gives the solidity of the materials; that is,
$68 \times 54{\cdot}96 \times 40 - 101832 = 47659{\cdot}2$ feet, the solidity required.

Note. When the arch A O B is an elliptical segment, its area multiplied by the length of the roof gives the solidity of the vacuity, and M L multiplied by S P, and the product by the length of the arch, gives the solidity of the cubic figure whose end is M N K L; and the difference of the two solidities is the solidity of the mixed solid whose section is A M N K L B E O H A. The materials of a bridge may be calculated after the same manner, by adding the solidities of T, T, and of the battlements, to the solidity as found in this Problem.

3. Required the capacity of the vacuity of an elliptical vault, whose span is 30 feet, and height 15 feet, the length of the vault being 90 feet? *Ans.* 31808·7 feet.

PROBLEM II.

To find the concave, or convex surface, of a circular, elliptical, or Gothic vaulted roofs.

Rule. Multiply the length of the arc by the length of the vault, and the product will be the superficies.

Note. To find the length of the arch, make a line ply close to it, quite across from side to side.

1. What is the surface of a vaulted roof, the length of the arch being 45 feet, and the length of the vault 140 feet?

$$140 \times 45 = 6300 \text{ square feet.}$$

2. Required the surface of a vaulted roof, the length of the arch being 40 feet 6 inches, and the length of the vault 100 feet? *Ans*. 40050 feet.

3. What is the surface of a vaulted roof, the length of the arch being 40·5 feet, and the length of the vault 60 feet? *Ans*. 2430 feet.

PROBLEM III.

To find the solidity of a dome, having the height and the dimensions of its base given.

RULE. Multiply the area of the base by the height, and $\frac{2}{3}$ of the product will give the solid content.*

1. What is the solid content of a dome, in the form of a hemisphere, the diameter of the circular base being 40 feet?

$$40^2 \times \cdot 7854 = 1256{\cdot}64 = \text{the area of base.}$$

$$\tfrac{2}{3}\,(1256{\cdot}64 \times 20) = \tfrac{2}{3}\,(25132{\cdot}8) = 16755{\cdot}2. \ \textit{Ans.}$$

2. What is the solid content of an octagonal dome, each side of its base being 20 feet, and the height 21 feet? *Ans*. 27039·1912 cubic feet.

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* This rule is correct only in one case, namely, when the dome is half a sphere, and in this case the height is equal to the radius of the circular base. It is a well-known property that the solidity of a sphere is $\frac{2}{3}$ of that of a cylinder having the same base and height. But the solidity of a cylinder is found by multiplying the area of its base by the height. Hence the reason of the rule, when applied to this particular case. No general rule can be given to answer every case, as some domes are circular, some elliptical, some polygonal, &c.; they are of various heights, and their sides of different curvature. When the height of the dome is equal to the radius of its base, (the curved sides being circular, or elliptical quadrants,) or to half a mean proportional between the two axes of its elliptical base, the above rule will answer pretty well; but with any other dimensions it ought not to be used.

3. Required the solidity of the stone-work of an elliptical dome, the two diameters of its base being 40 and 30 feet, the height 17·32 feet, and the stone-work in every part 4 feet thick? *Ans.* 9479·086848 cubic feet.

PROBLEM IV.

To find the superficial content of a dome, the height and dimensions of its base being given.

RULE. When the base is circular, multiply the square of the diameter of the base by 1·5708, and the product will be the superficial content.

For an elliptical dome, multiply the two diameters of its base together, and the product resulting by 1·5708 for the superficial content, sufficiently correct for practical purposes.*

1. The diameter of the base of a circular dome is 20 feet, and its height 10 feet; required its concave superficies?

$20^2 \times 1.5708 = 628{\cdot}32$ feet, the *Ans.*

2. The two diameters of an elliptical dome are 40 and 30 feet, and its height 17·32 feet; required the concave surface? *Ans.* 1884·96 square feet.

3. What is the superficies of a hexagonal spherical dome, each side of the base being 10 feet? *Ans.* 519·6152.

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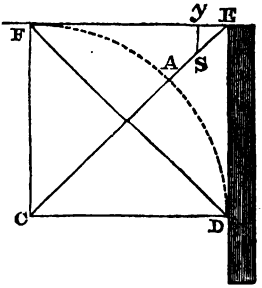
* The same objection applies to this as to the last rule, being correct only when the dome is circular, and its height equal to the radius of the base, Because the surface of a sphere is equal to the curved surface of its circumscribing cylinder; but the curved surface of a cylinder whose diameter is D, and height $\frac{D}{2}$ has been shown to be $3{\cdot}1416\,D \times \frac{D}{2} = 1{\cdot}5708\,D^2$. Hence the reason of the rule.

PROBLEM V.

To find the solid content of a saloon.

RULE. Multiply the area of a transverse section by the compass or circumference of the solid part of the saloon, taken round the middle part. Subtract this product from the whole vacuity of the room, supposing the walls to go upright from the spring of the arch to the flat ceiling, and the difference will be the answer.

1. What is the solid content of a saloon with a circular quadrantal arch of 2 feet radius, springing over a rectangular room of 20 feet long and 16 feet long.



$2^2 \times \cdot 7854 = 3 \cdot 1416$ = area of the quadrant C D A E. $2 \times 2 \div 2 = 2$ = area of the triangle C D E; then $3 \cdot 1416$

* $CD^2 \times \cdot 7854 = 3 \cdot 1416$ the area of the quadrant C D A E; and $\frac{1}{2}$ (C D × C E) = 2 the area of the triangle C D E; therefore $3 \cdot 1416 - 2 = 1 \cdot 1416$ is the area of the segment D A E. Now C D × C E = 4 the area of the rectangle C D F E; therefore $4 - 3 \cdot 1416 = \cdot 8584$ the area of the section D F E A D. Again, $\sqrt{(CD^2 + CE^2)}$ = D E = $2 \cdot 8284271$. The compass of the room will be equal to twice the length added to twice the breadth; that is, $20 \times 2 + 16 \times 2$, 72 feet, which evidently exceeds the circumference of the middle of the solid projection of the saloon. Because $\frac{1}{2}$ (C F − C A) = $\frac{1}{2}$ ($2 \cdot 8284271 - 2$) = $\cdot 4142136$ = F S, and C F : F S :: F E : F y = $\cdot 2928932$; hence $72 - (\cdot 2928932 \times 8) = 69 \cdot 6568544$ = the solidity of the middle of the solid part of the saloon. The rest is evident from the text.

$-2 = 1{\cdot}1416 =$ area of the segment D A E. Now, $2 \times 2 = 4 =$ area of the rectangle C D F E; then $4 - 3{\cdot}1416 = {\cdot}8584 =$ area of the section D F E A D. $\sqrt{(2^2 + 2^2)} = \sqrt{8} = 2{\cdot}8284271$. $2 \times 16 + 2 \times 20 = 72 =$ the compass within the walls. $\frac{1}{2}(2{\cdot}8284271 - 2) = 4142136 =$ F S and $2{\cdot}8284271 : {\cdot}4142136 :: 2 : {\cdot}2928932 =$ F y : hence $72 - ({\cdot}2928932 \times 8) = 69{\cdot}6568544 =$ the circumference of the middle of the solid part of the saloon; therefore $69{\cdot}6568544 \times {\cdot}8584 = 59{\cdot}79344381696 =$ the content of the solid part of the saloon.

$20 \times 16 = 320$ the area of the room floor, and $320 \times 2 = 640 =$ the solidity of the upper part of the room; then $640 - 59{\cdot}79344 = 580{\cdot}20656$ feet, the solidity of the saloon.

2. If the height D F of the saloon be 3·2 feet, the chord D E $= 4{\cdot}5$, and A V $= 9$ inches; what is the solid content of the solid part, the mean compass being 50 feet?

Ans. 137·9.

PROBLEM VI.

To find the superficies of a saloon.

RULE. Find its breadth by applying a string close to it across the surface; find also its length by measuring along the middle of it, quite round the room; then multiply these two dimensions together for the superficial content.

1. The girt across the face of the saloon is 5 feet, and its mean compass 100 feet; what is its superficial content?

$100 \times 5 = 500$, the answer.

2. The girt across the face of the saloon is 12 feet, and its mean compass 98; required its surface?

Ans. 1176 feet.

SPECIFIC GRAVITY.

SECTION IX.

1. The specific gravity of a body is the relation which the weight of a given magnitude of that body has to the weight of an equal magnitude of a body of another kind.

In this sense a body is said to be specifically heavier than another, when under the same bulk it weighs more than that other.

On the contrary, a body is said to be specifically lighter than another, when under the same bulk it weighs less than that other. Thus, if there be two equal spheres, each one foot or one inch in diameter, the one of lead and the other of wood, then since the leaden sphere is found to be heavier than the wooden one, it is said to be specifically, or in specie, heavier, and the wooden sphere specifically lighter.

2. If two bodies be equal in bulk, their specific gravities are to each other as their weights, or as their densities.

3. If two bodies be of the same specific gravity or density, their absolute weights will be as their magnitudes or bulks.

4. If two bodies be of the same weight, the specific gravities will be reciprocally as their bulks.

5. The specific gravities of all bodies are in a ratio compounded of the direct ratio of their weights, and the reciprocal ratio of their magnitude. Hence again the specific gravities are as the densities.

6. The absolute weights or gravities of bodies are in the compound ratio of their specific gravities and magnitudes or bulks.

7. The magnitudes of bodies are directly as their weights, and reciprocally as their specific gravities.

8. A body specifically heavier than a fluid, loses as much of its weight, when immersed in it, as is equal to the weight of a quantity of the fluid of the same bulk or magnitude; if the body be of equal density with the fluid, it loses all its weight, and requires no force but the fluid to sustain it. If it be heavier, its weight in the fluid will be only the difference between its own weight and the weight of the same bulk of the fluid; and therefore it will require a force equal to this difference to sustain it. But if the body immersed be lighter than the fluid, it will require a force equal to the difference between its own weight and that of the same bulk of the fluid, to keep it from rising in the fluid.

9. In comparing the weights of bodies, it is necessary to consider a certain quantity of some one as the unit standard with which all other bodies may be compared. Rain water is generally taken as the standard, it being found to be nearly alike in all places.

A cubic foot of rain-water is found, by repeated experiments, to weigh $62\frac{1}{2}$ pounds avoirdupoise, or 1000 ounces, and a cubic foot containing 1728 cubic inches, it follows that a cubic inch weighs ·03616898148 pound. Therefore if the specific gravity of any body be multiplied by ·03616898148, the product will be the weight of a cubic inch of that body in pounds avoirdupois; and if this weight be multiplied by 175, and the product be divided by 144, the quotient will be the weight of a cubic inch in pounds troy, 144 pounds avoirdupois being exactly equal to 175 pounds troy.

10. Since the specific gravities of bodies are as their absolute gravities under the same bulk; the specific gravity of a fluid will be to the specific gravity of any body immersed in it, as the part of the weight lost by the solid, is to the whole weight. Hence the specific gravities of different fluids are as the weights lost by the same solid immersed in them.

PROBLEM I.

To find the specific gravity of a body.

CASE I. *When the body is heavier than water.*

Weigh the body first in water, and afterwards in the open air, the difference will give the weight lost in water; then say, as the weight lost in water is to the absolute weight of the body, so is the specific gravity of water to the specific gravity of the body.

CASE II. *When the body is lighter than water.*

Fix another body to it, so heavy as that both may sink in water together, as a compound mass. Weigh the compound mass and the heavier body separately, both in the water and open air, and find how much each loses in water, by taking its weight in water from its weight in the open air. Then say, as the difference of these remainders, is to the weight of the lighter body in air; so is the specific gravity of water, to the specific gravity of the lighter body.

CASE III. *For a fluid of any kind.*

Weigh a body of known specific gravity both in the fluid and open air, and find the loss of weight, by subtracting the weight in water from the weight out of it. Then say, as the whole, or absolute weight, is to the loss of weight; so is the specific gravity of the solid, to the specific gravity of the fluid.

The usual way of finding the specific gravities of bodies is the following, viz.

On one arm of a balance suspend a globe of lead by a fine thread, and to the other arm of the balance fasten an equal weight sufficient to balance it in the open air; immerse the globe into the fluid, and observe what weight balances it then, by which the lost weight is ascertained, which is proportional to the specific gravity.

Immerse the globe successively in all the fluids whose proportional specific gravity you require, observing the

weight lost in each; then these weights lost in each will be the proportions of the fluids sought.

1. A piece of platina weighed 83·1886 pounds out of water, and in water only 79·5717 pounds; what is its specific gravity; that of water being 1000?

83·1886 — 79·5717 = 3·6169 pounds, which is the weight lost in water.

Then 3·6169 : 83·1886 :: 1000 : 23000, the specific gravity, or the weight of a cubic foot of metal in ounces.

2. A piece of stone weighed 10lb in the open air, but in water only $6\frac{3}{4}$lb; what is its specific gravity?

Ans. 3077.

Examples to Case II.

3. If a piece of elm weigh 15lb in the open air; and that a piece of copper, which weighs 18lb in air, and 16lb in water, is affixed to it, and that the compound weighs 8lb in water; required the specific gravity of the elm?

Copper.	Compound.
18 in air.	33
16 in water.	6
—	—
2 loss.	27
	2
	—
	As 25 : 15 :: 1000 : 600th;

specific gravity of the elm.

4. A piece of cork weighs 20 lb in air, and a piece of granite being fixed to it, which weighs 120 lb in air, and only 80 lb in water, the compound mass weighs $16\frac{3}{4}$ lb in water; required the specific gravity of the cork? *Ans.* 240.

Examples to Case III.

5. A piece of cast-iron weighed 259·1 ounces in a fluid, and 298·1 ounces out of it; required the specific gravity of the fluid, allowing the specific gravity of the cast-iron to be 7645?

298·1 — 259·1 = 39 loss of weight in the iron; then

298·1 : 39 :: 7645 : 1000, the specific gravity of the fluid; showing the fluid to be water.*

6. A piece of lignum vitæ weighed $42\frac{3}{4}$ ounces in a fluid, and $166\frac{5}{8}$ ounces out of it; what is the specific gravity of the fluid; that of the lignum vitæ being 1333?

Ans. 991 is the specific gravity of the fluid, which shows it to be liquid turpentine.

TABLE OF SPECIFIC GRAVITIES.

	Spec. grav.	wt. cub. in. oz.
Platina	19500	11·285
Do. hammered	20336	11·777
Cast zinc	7190	4·161
Cast iron	7207	4·165
Cast tin	7291	4·219
Bar iron	7788	4·507
Hard steel	7816	4·523
Cast brass	8395	4·858
Cast copper	8788	5·085
Pure cast silver	10474	6·061
Cast lead	11352	6·569
Mercury	13568	7·872
Pure cast gold	19258	11·145
Amber	1078	wt. cub. ft.
Brick	2000	125·00
Sulphur	2033	127·06
Cast nickel	7807	4513
Cast cobalt	7811	4520
Paving stones	2416	151·00
Common stone	2520	157·50
Flint and spar	2594	162·12
Green glass	2642	
White glass	2892	
Pebble	2664	166·50
Slate	2672	167·00
Pearl	2684	
Alabaster	2730	
Marble	2742	171·38
Chalk	2784	174·00
Limestone	3179	193·68
Wax	897	
Tallow	945	

---

* In this manner may the species of a fluid or solid be ascertained, by means of its specific gravity, and the Table at the end of this chapter, exhibiting the specific gravities of various bodies. This Table has been taken from Gregory's work for Practical Men.

	Spec. grav.		wt. cub. ft. lbs.
Camphor	989		
Bees' wax	965		
Honey	1456		
Bone of an ox	1659		
Ivory	1822		
Air at the earth's surface	14		
Oil of turpentine	870		
Olive oil	915		
Burgundy	991		
Distilled water	1·000		
Sea water	1·028		
Milk	1·030		
Beer	1·034		
Cork	240	....	15·00
Poplar	383	....	23·94
Larch	544	....	34·00
Elm, and West India Fir	556	....	34·75
Mahogany	560	....	35·00
Cedar	596	....	37·25
Pitch pine	660	....	41·25
Pear tree	661	....	41·31
Walnut	671	....	41·94
Elder tree	695	....	43·44
Beech	696	....	43·50
Cherry tree	715	....	44·68
Maple and Riga fir	750	....	46·87
Ash and Dantzic oak	760	....	47·50
Apple tree	793	....	49·56
Alder	800	....	50·00
Oak, Canadian	872	....	54·50
Box, French	912	....	57·00
Logwood	913	....	57·06
Oak, English	970	....	51·87
Oak, 60 years old	1170	....	73·12
Ebony	1331	....	83·18
Lignum vitæ	1333	....	83·31

PROBLEM II.

The specific gravity of a body, and its weight, being given, to find its solidity.

RULE. Say, as the tabular specific gravity of the body, is to its weight, in ounces avoirdupois, so is 1 cubic foot, 1728 inches, to the content in feet, or inches.

1. What is the solidity of a block of marble, that weighs 10 *tons;* its specific gravity being 2742 ?

First, 10 tons = 200 hundreds = 800 quarters = 89600 pounds = 1433600 ounces; then

$$
\begin{array}{l}
2742 : 1433600 :: 1 \text{ cubic foot} \\
\qquad\qquad\quad 1 \\
\hline
2742)1433600(522\tfrac{3}{4} \\
\quad 13710 \\
\hline
\quad\;\; 6260 \\
\quad\;\; 5484 \\
\hline
\quad\;\;\; 7760 \\
\quad\;\;\; 5484 \\
\hline
\end{array}
$$

$$\frac{2276}{2742} = \tfrac{3}{4} \text{ nearly.}$$

2. How many cubic inches in an irregular block of marble which weighs 112 pounds, allowing its specific gravity to be 2520? *Ans.* $1228\frac{2016}{2520}$ cubic inches.

3. How many cubic inches of gun-powder are there in 1 pound weight, its specific gravity being 1745? *Ans.* $38\frac{3}{4}$ feet nearly.

3. How many cubic feet are there in a ton weight of dry oak, its specific gravity being 925? *Ans.* $38\frac{138}{185}$.

PROBLEM III.

The linear dimensions, or magnitude of a body, being given, and also its specific gravity, to find its weight.

RULE. One cubic foot, or 1728 cubic inches, is to the solidity of the body, in inches; as the tabular specific gravity of the body, is to the weight in avoirdupois ounces.

1. What is the weight of a piece of dry oak, in the form of a parallelopipedon, whose length is 58 inches, breadth 18 inches, and depth 12?

$65 \times 18 \times 12 = 12096$ cubic inches, the solid content.

Then $1728 : 12096 :: 932 : 6524$ ounces $= 407\frac{3}{4}$ pounds, the weight required.

2. What is the weight of a block of dry oak, which measures 10 feet long, 3 feet broad, and $2\frac{1}{2}$ feet deep?

Ans. $4335\frac{15}{16}$.

3. What is the weight of a block of marble, whose length is 63 feet, and its breadth and thickness, each 12 feet?

Ans. $683\frac{7}{16}$.

PROBLEM IV.

To find the quantities of two ingredients in a given compound.

RULE. Take the difference of every pair of the three specific gravities, viz. of the compound and each ingredient; and multiply the difference of every two by the third.

Then as the greater product is to the whole weight of the compound, so is each of the other products to the weights of the two ingredients.

1. A composition of 112 lb being made of tin and copper, whose specific gravity is found to be 8784; what is the quantity of each ingredient, the specific gravity of tin being 7320, and of copper 9000?

9000	9000	8784	
7320	8784	7320	
1680	216	1464	diff.
8784	7320	9000	
14757120	1581120	13176000.	Then

$$14757120 : 112 :: \begin{cases} 13176000 : 100\text{ lb copper.} \\ 1581120 : 12\text{ lb tin.} \end{cases}$$

2. Hiero, king of Sicily, furnished a goldsmith with a quantity of gold, to make a crown. When it came home, he suspected that the goldsmith had used a greater quantity of silver than was necessary in the composition; and applied to the famous mathematician, Archimedes, a Syracusian, to discover the fraud, without defacing the crown.

To ascertain the quantity of gold and silver in the crown, he procured a mass of gold and another of silver, each exactly of the same weight with the crown; justly considering

that if the crown were of pure gold, it would be of equal bulk, and therefore displace an equal quantity of water with the golden mass; and if of silver, it would be of equal bulk, and displace an equal quantity of water with the silver mass; but if of a mixture of the two, it would displace an intermediate quantity of water.

Now suppose that each of the three weighed 100 ounces; and that on immersing them severally in water, there were displaced 5 ounces of water by the golden mass, 9 ounces by the silver mass, and 6 ounces by the crown; what quantity of gold and silver did the crown contain?

Ans. { 75 ounces of gold.
25 ounces of silver.

Note. For the reason of the rule given above, and for the solution of this question, see Question 21, Alligation total, in Arithmetic for the use of the Irish Education Schools; or Gregory's Philosophy of Arithmetic.

Questions relating to specific gravities may be wrought by the rules of Alligation, as well as by any Algebraic process that might be employed.

PROBLEM V.

To find how many inches a floating body will sink in a fluid.

RULE. Find, by Problem II., the weight of the floating body from its solidity and specific gravity, and that will be the weight of the fluid which it will displace.

Then say, as the specific gravity of the fluid is to 1728 cubic inches; so is the weight of the body, in ounces, to the cubic inches immersed. The depth will be found from the given dimensions.

1. Suppose a piece of dry oak, in the form of a parallelopipedon, whose length is 56 inches, breadth 18, and depth 12, is to be floated upon common smooth water, on its broadest side; how many inches will it sink, its specific gravity being 932?

By Problem II. the weight of the piece of oak is 6524 ounces, which by the preliminary part of this chapter, is the weight of water displaced.

Then 1000 : 1728 : : 6524 : 11273·472 cubic inches of oak immersed.

Therefore 11273·472 ÷ 56 × 18 = 11·184 inches the depth it will sink.

To find how far it will sink, allowing it to float on its narrower side 11273·472 ÷ 56 × 12 = 16·776 inches.

2. How many inches will a cubic foot of dry oak sink in common water, allowing the specific gravity of the oak to be 970? *Ans.* 9·95904.

PROBLEM VI.

To find what weight may be attached to a floating body, so that it may be just covered with fluid.

RULE. Multiply the cubic feet in the body by the difference between its specific gravity and that of the fluid, and the product will be the weight in ounces avoirdupois, just sufficient to immerse it in the fluid.

1. What weight must be attached to a piece of dry oak, 56 inches long, 18 inches broad, and 12 inches deep, will keep it from rising above the surface of a fresh water lake; the specific gravity of the water being 1000, and that of the oak 932?

Here 56 × 18 × 12 = 12096 cubic inches.

Then 12096 ÷ 1728 = 7 feet.

Then (1000 − 932) × 7 = 68 × 7 = 476 ounces = 29 pounds 12 ounces.

2. What weight, fixed to a piece of dry oak, 9 inches long, 6 inches broad, and 3 inches deep, will keep it from rising above the surface of common water, the specific gravity of water being 1000, and that of the oak 970?

Ans. $2\frac{13}{16}$ ounces.

3. A sailor had half an anker of brandy, the specific gravity of the liquor was 927, the cask was oak, and contained 216 cubic inches, and its specific gravity was 932; to secure his prize from the custom-house officers, he fixed just as

much lead to the cask as would keep it under water, and then threw it into the sea; what weight of lead was necessary for his purpose?

Ans. The cask of brandy contained 1371 cubic inches, the weight of sea-water of an equal bulk was 817·20486 ounces, the cask weighed 116·5 ounces, the brandy 619·609375, both together weighed 736·19375 ounces. The difference between the specific gravity of lead and sea-water is to this remainder, as the specific gravity of lead to its weight in ounces, which will be fond to be 89·09495 ounces, or 5 pounds 9 ounces.

PROBLEM VII.

To find the solidity of a body, lighter than a fluid, which will be sufficient to prevent a body much heavier than the fluid, from sinking.

RULE. Find the solidity of the body to be floated; from its weight and specific gravity, by Problem Find also the weight of an equal bulk of the fluid by Problem II. Then say, as the difference between the specific gravity of the fluid, and that of the body lighter than the fluid, is to the difference between the weight of the body to be floated and the weight of an equal bulk of the fluid, so is 1728 to the solidity of the lighter body, in cubic inches.

1. How many solid feet of yellow fir, whose specific gravity is 657, will be sufficient to keep a brass cannon, weighing 56 cwt., a-float at sea, the specific gravity of brass being 8396, and of sea-water 1030?

First, 56 cwt. = 100352 ounces, weight of the body to be floated.

Then, 8396 : 100352 :: 1728 : 20653·675 cubic inches in the cannon.

And, 1728 : 20653·675 :: 1030 : 12310·9289, the weight of sea-water equal in bulk to that of the cannon.

Hence, 1030 — 657 : 100352 — 12310·9289 :: 1728 : 407868·5545 cubic inches = 236·036 feet, the *Ans.*

2. The specific gravity of lead is 11325, of cork 240, and

N

of sea-water 1030; now it is required to know how many cubic inches of cork will be sufficient to keep $49\frac{3}{8}$ pounds of lead a-float at sea? *Ans.* 1570·84 cubic inches.

TO FIND THE TONNAGE OF SHIPS.

1st.—VESSELS A-GROUND.

By the Parliamentary rule.

PROBLEM VIII.

The length of the ship is to be measured on a straight line along the lower edge of the rabbet of the keel, from a perpendicular, or square, from the back of the main post, at the height of the wing transom, to a perpendicular at the height of the upper deck, (but the middle deck of three-decked ships,) from the fore-part of the stem; then from the length between these perpendiculars subtract $\frac{3}{5}$ of the extreme breadth for the rake of the stem, and $2\frac{1}{2}$ inches for every foot of the height of the wing transom above the lower part of the rabbet of the keel, for the rake abaft; and the remainder will be the length of the keel for tonnage.

The main breadth is to be taken from the outside of the outside plank, in the broadest part of the ship, either above or below the wales, deducting therefrom all that it exceeds the thickness of the plank of the bottom, which shall be accounted the main breadth; so that the moulding breadth, or the breadth of the frame, will then be less than the main breadth, so found, by double the thickness of the plank of the bottom.

Then multiply the length of the keel for tonnage, by the main breadth, so taken, and the product by half the breadth, then divide the whole by 94, and the quotient will give the tonnage.

In cutters and brigs, where the rake of the stern-post exceeds $2\frac{1}{2}$ inches to every foot in height, the actual rake is

generally subtracted instead of the $2\frac{1}{2}$ inches to every foot, as before mentioned.

1. Let us suppose the length from the fore-part of the stern, at the height of the upper deck, to the after-part of the stern-post, at the height of the wing transom, to be 155 feet 8 inches, the breadth from out to outside 40 feet 6 inches, and the height of the wing transom 21 feet 10 inches; what is the tonnage?

$$
\begin{array}{lrl}
 & 40{\cdot}6 & \text{breadth} \\
\text{deduct} & 3 & \\
\hline
 & 40{\cdot}3 & \\
 & 3 & \\
\hline
 & 5)120{\cdot}9 & \\
\hline
 & 24{\cdot}1\frac{4}{5} = 24{\cdot}15 & \\
 & 21{\cdot}10 & \text{height of wing transom.} \\
 & 2\frac{1}{2} & \text{multiply.} \\
\hline
 & 12)54\frac{7}{12} &
\end{array}
$$

$$4{\cdot}55 + 24{\cdot}15 = 28{\cdot}70$$

$$155{\cdot}66 - 28{\cdot}70 = 126{\cdot}96 = \text{length.}$$

$$\frac{126{\cdot}96 \times 40{\cdot}25 + 20{\cdot}125}{94} = 1094 \text{ the } \textit{Ans.}$$

2. Suppose the length of the keel to be 50·5 feet, breadth of the midship-beam 20 feet; required the tonnage?

Ans. 107·4.

3. If the length of the keel be 100 feet, and the breadth of the beam 30 feet; what is the tonnage? *Ans.* 478.

2nd.—VESSELS A-FLOAT.

Drop a plumb-line over the stern of the ship, and measure the distance between such line and the after-part of the stern-post, at the low-water mark: in a parallel direction with the water, to a perpendicular point immediately over the low-water mark, at the fore-part of the main-stern, subtracting from such measurement the above distance, the remainder will be the ships extreme length; from which is

to be deducted three inches for every foot of the load draught of water for the rake abaft, and also three-fifths of the ship's breadth for the rake forward, the remainder shall be esteemed the just length of the keel to find the tonnage; and the breadth shall be taken from outside to outside of the plank in the broadest part of the ship, either above or below the main-wales, exclusive of all manner of sheathing or doubling that may be wrought upon the sides of the ship; then multiply the length of the keel, taken as before directed, by the breadth, as before taught, and that product by half the said breadth, and dividing the product by 94, the quotient is the tonnage.

3rd.—STEAM VESSELS.

The length shall be taken on a straight line, along the rabbet of the keel of the ship, from the back of the main-stern-post to a perpendicular line from the fore-part of the main-stern under the bow-sprit; from which deducting the length of the engine-room, and subtracting three-fifths of the breadth, the remainder shall be esteemed the just length of the keel to find the tonnage; and the breadth shall be taken from the outside of the outside plank in the broadest place of the ship or vessel, be it either above or below the main-wales, exclusively of all manner of doubling planks that may be wrought upon the sides of the ship or vessel; then multiply the length and breadth so found together, and that product by half the same breadth, and dividing by 94, the quotient will be the tonnage, according to which all such vessels shall be measured.

Note. Under certain penalties nothing but the fuel can be stowed in the engine-room.

Some divide the last product by 100, to find the tonnage of king's ships, and by 95, to find that of merchants' ships.

FLOATING BODIES.

1. The buoyancy of casks, or the load which they will carry without sinking, may be estimated, by reckoning 10 lb avoirdupois to the ale gallon, or $8\frac{1}{5}$ lb to the wine gallon.

2. The buoyancy of pantoons may be estimated at about *half a* hundred weight, or 56 lb for each cubic foot. There-

fore a pantoon which contained 96 cubic feet, would sustain 48 hundred weight before it would sink.

N. B.—This is an approximation, in which the difference between $\frac{6}{11}$ and $\frac{1}{2}$, viz. $\frac{1}{22}$ of the whole weight is allowed for that of the pantoon itself.

3. The principles of buoyancy are very ingeniously applied in the self-acting flood-gate, which, in the case of common sluices to a mill-dam, prevents inundation when a sudden flood occurs. By means of the same principle, it is, that a hollow ball attached to a metallic lever of about a foot long, is made to rise with the liquid in a water-cask, and thus to close the cock and stop the supply from the pipe, just before the time when the water would otherwise run over the top of the vessel.

The property of buoyancy has also been successively employed in raising ships which had sunk under water, and in pulling up old piles in a river when the tide ebbs and flows. A large barge is brought over a pile as the water begins to rise; a strong chain which has been previously fixed to the pile by a ring, &c. is made to gird the barge, and is then firmly fastened; then, as the tide rises the barge rises also, and by means of its buoyant force draws up the pile with it.

In a case which actually occurred, a barge of 50 feet long, 12 feet wide, 6 deep, and drawing two feet water, was employed. Then $50 \times 12 \times (6-2) \times \frac{4}{7} = \frac{50 \times 12 \times 16}{7}$ $= 192 \times 7\frac{1}{7} = 1344 + 27\frac{3}{7} = 1371\frac{3}{7}$ cwt. $= 66\frac{1}{2}$ tons, nearly, which is the measure of the force with which the barge acted in pulling up the pile.

WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

SECTION X.

The foregoing Problems furnish rules for finding the weight and dimensions of balls and shells. But they may be found much easier by means of the experimental weight of a ball of a given size, and from the well-known geometrical property, that similar solids are as the cubes of their diameters.

PROBLEM I.

To find the weight of an iron ball, from its diameter.

RULE. Nine times the cube of the diameter being divided by 64, will express the required weight in pounds.*

1. The diameter of an iron shot is 5 inches; required its weight.

$5 \times 5 \times 5 = 125 =$ cube of the ball's diameter.

Then, $125 \times 9 \div 64 = 17\frac{37}{64}$ lbs, the *Ans.*

2. The diameter of an iron shot being 3 inches; required its weight? *Ans.* 3·8 lb.

3. The diameter of an iron shot is 5·54 inches; what is its weight? *Ans.* 24 lb.

---

* It is known by experiment that an iron ball of 4 inches in diameter weighs 9lb; and the weights of bodies composed of the same materials being as their quantities of matter; that is, as the cubes of their diameters. It will be, as 4^3 (64) : the cube of any other ball :: 9lb : *the weight* required, which affords the rule.

PROBLEM II.

To find the weight of a leaden ball, by having its diameter given.

RULE. Multiply the cube of its diameter by 2, and divide the product by 9, and the quotient will give the weight in pounds.*

1. What is the weight of a leaden ball of 5 inches diameter?

$$5 \times 5 \times 5 = 125 = \text{cube of ball's diameter.}$$

$$\text{Then, } 125 \times 2 \div 9 = 250 \div 9 = 27\tfrac{7}{9}\text{lb, } \textit{Ans.}$$

2. What is the weight of a leaden ball whose diameter is 6·6 inches? *Ans.* 63·888 lb.

3. What is the weight of a leaden ball whose diameter is 3·5 inches? *Ans.* 9·53 lb.

4. What is the weight of a leaden ball whose diameter is 6 inches? *Ans.* 48 lb.

PROBLEM III.

Having the weight of an iron ball, to determine its diameter.

RULE. Multiply the weight by $7\frac{1}{9}$, then take the cube root of the product for the diameter.†

1. What is the diameter of an iron ball, whose weight is 42 lb?

$$42 \times 7\tfrac{1}{9} = 298\tfrac{2}{3}.$$

$$\text{Then, } \sqrt[3]{298} = 6{\cdot}685 \text{ inches, the } \textit{Ans.}$$

2. Required the diameter of an iron ball, whose weight is 24 lb? *Ans.* 5·54 inches.

---

* It is found by experiment that a leaden ball of $4\frac{1}{4}$ inches diameter weighs 17lb. But the cube of $4\frac{1}{4}$ is to 17 nearly as 9 to 2. Hence and from the proportion, that similar solids are as the cubes of their diameters, the rule is obvious.

† This rule is obvious from Prob. I., being the converse thereof.

3. What is the diameter of an iron ball, whose weight is 3·8 lb? *Ans.* 3 inches.

PROBLEM IV.

Having the weight of a leaden ball, to determine its diameter.

RULE. Multiply the weight by 9, and divide the product by 2; and the cube root of the quotient will express the diameter.*

1. What is the diameter of a leaden ball, whose weight is 64 lb?

$$64 \times 9 = 576.$$

$$\text{Then, } 576 \div 2 = 288.$$

$$\text{Hence, } \sqrt[3]{288} = 6{\cdot}6 \text{ inches, the } Ans.$$

2. Required the diameter of a leaden ball, whose weight is $27\frac{7}{9}$ lb? *Ans.* 5 inches.

3. What is the diameter of a leaden ball, whose weight is 63·888 lb. *Ans.* 6·6 inches.

PROBLEM V.

Having given the external and internal diameters of an iron shell, to find its weight.

RULE. Find the difference between the cubes of the two diameters, and multiply it by 9; divide the product by 64, and the quotient will express the weight in pounds.†

1. What is the weight of an 18 inch iron bomb-shell, whose mean thickness is $1\frac{1}{4}$ inches?

---

* This rule is manifest from Problem II., being only the converse thereof.

* The reason of this rule may be easily derived from Problem I. For, by deducting the weight sufficient to fill the cavity, from the capacity of the external surface, the remainder will express the weight of the shell. Let d and D express the internal and external diameters; then their weights will be $d^3 \times \frac{9}{64}$ and $D^3 \times \frac{9}{64}$.·. their *difference*, viz. $(D^3 - d^3) \times \frac{9}{64}$ will give the weight of the shell.

$18 - 2\frac{1}{2} = 15\frac{1}{2}$ = internal diameter.

Then, $18^3 = 5832$ the cube of external diameter.

$(15\cdot5)^3 = 3723\cdot875$ the cube of internal diameter.

And, $5832 - 3723\cdot875 = 2108\cdot125$ = difference of cubes.

Hence, $2108\cdot125 \times 9 \div 64 = 296\cdot45$lb, the *Ans.*

2. What is the weight of a 9 inch iron bomb-shell, whose mean thickness is $1\frac{1}{2}$ inch? *Ans.* 72·14lb.

3. What is the weight of an iron bomb-shell, whose external diameter is 9·8 inches, and internal diameter 7 inches? *Ans.* $84\frac{1}{8}$lb.

PROBLEM VI.

To find how much powder will fill a shell of given dimensions.

RULE. Divide the cube of the internal diameter in inches, by 57·3, and the quotient will express the answer.*

1. What quantity of powder will fill a shell, whose internal diameter is 10 inches?

First, $10 \times 10 \times 10 = 1000$ = cube of diameter.

```
57·3)1000(17·45lb, Ans
     573
     ----
     4270
     4011
     ----
      2590
      2292
      ----
       2980
       2865
       ----
        115, &c.
```

* By experiment it is found that 1 lb avoirdupois weight, of gunpowder contains 31·06 cubic inches. And putting d for the internal diameter, the capacity of the shell is $d^3 \times \cdot5236$. Hence $31\cdot06 : d^3 \times \cdot5236 :: 1$ lb : is the weight required in pounds; that is, $d^3 \times \cdot5236 \div 31\cdot06 = \frac{d^3}{59\cdot32}$, which is the rule, according to the note.

The rule in the text may be explained in a similar manner.

Note. In some recent works, the cube of the diameter is divided by 59·32 for the weight of powder in pounds.

2. How many pounds of gunpowder are required to fill a hollow shell, whose internal diameter is 13 inches?
Ans. 37 lb, according to the note.

3. Required the number of pounds of powder that will fill a shell, whose internal diameter is 7 inches?
Ans. 6 lb, by the rule in the text.

PROBLEM VII.

To find how much powder will fill a rectangular box of given dimensions.

RULE. Multiply the length, breadth, and depth together, in inches, and the last result by ·0322, and the last product will give the weight in pounds.*

1. How many pounds of powder will fill a rectangular box, whose length is 16 inches, breadth 12 inches, and depth 6 inches?

$$16 \times 12 \times 6 = 1152 = \text{content of the box.}$$

$$\text{Then, } 1152 \times {\cdot}0322 = 37{\cdot}0944 \text{ lb, the } Ans.$$

2. How many pounds of powder will fill a rectangular box, whose length is 10 inches, breadth 5 inches, and depth 2 inches?
Ans. 3·22 lb.

3. How many pounds of powder will fill a rectangular box, whose length is 5 inches, breadth 2 inches, and depth 10 inches?
Ans. 3·22 lb.

---

* Let l, b, and d, represent the length, breadth, and depth, respectively; then the content will be $l\,b\,d$; then, as in the last, it will be, as $31{\cdot}06 : l\,b\,d :: 1$ lb : the weight required; that is, $\frac{l\,b\,d}{31{\cdot}06} = l\,b\,d \times {\cdot}0322$, which is the rule.

PROBLEM VIII.

Having the length and diameter of a cylinder, to determine how many pounds of gunpowder will fill it.

RULE. Multiply the square of the diameter by the length, and divide the product by 40, for the weight in pounds.*

1. The diameter of a hollow cylinder is 10 inches, and the length 14 inches; how many pounds will it hold?

$$10 \times 10 = 100 = \text{square of diameter.}$$
$$\text{Then, } 100 \times 14 = 1400.$$
$$\text{Hence, } 1400 \div 40 = 35\text{lb, the } \textit{Ans.}$$

2. The diameter of a hollow cylinder is 5 inches, and its length 40 inches; how much powder will it hold?

Ans. 25lb.

3. The diameter of a hollow cylinder is 5 inches, and the length 12 inches; how many pounds will it hold?

Ans. 7·5lb.

PROBLEM IX.

To find what portion of a cylinder will be occupied by a given quantity of powder, the diameter of the cylinder being given.

RULE. Multiply the given weight of powder by 40, and divide the product by the square of the diameter of the cylinder, and the quotient will be the pounds required.†

---

* Put d equal the diameter of the cylinder, and l for the length; then its content is $d^2\,l \times {\cdot}7854$; then, as in the two last $31{\cdot}06 : d^2\,l \times {\cdot}7854 :: 1\text{ lb} : \text{the weight}$; that is, $d^2\,l \times {\cdot}7854 \div 31{\cdot}06 = d^2\,l \div 40$ nearly; which is the rule.

† Retaining the same notation, as in the last, and putting $w =$ the weight, we have $w = \frac{d^2\,l}{40}$; then $40\,w = d^2\,l$; divide both sides of the equation by d^2, and we get $l = \frac{40\,w}{d^2}$, which is the rule.

Note. The foregoing rules only approximate the truth.

1. The diameter of a hollow cylinder is 10 inches; how much of it will hold 50lb of powder?

$$50 \times 40 = 2000.$$

Then $2000 \div 100 = 20$ inches, the *Ans.*

2. How much of a cylinder of 14 inches diameter, will hold 10lb of powder? *Ans.* 2·05 inches.

3. How much of a cylinder, 12 inches in diameter, will hold 144lb of powder? *Ans.* 40 inches.

PILING OF BALLS AND SHELLS.

Iron-shot and shells are usually piled in horizontal courses, either in a pyramidical or in a wedge-like form; the base being either an equi-lateral triangle, a square, or a rectangle.

Those piles whose bases are triangles, or squares, terminate in one ball at the top; but piles whose bases are rectangles terminate in a single row of balls.

In triangular and square piles, the number of horizontal rows, or courses, is always equal to the number of balls in one side of the bottom row.

And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom.

Also the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row.

PROBLEM I.

To find the number of balls in a rectangular pile.

RULE. Multiply together, the number in one side of the bottom row, that number increased by 1; and the same number increased by 2; then the one-sixth of the last product will give the number of balls required.*

---

* The reason of this rule is derived from the method for finding the sum of a triangular progression. Thus, in a triangular pile it is *obvious that* each course of balls is in the shape of a triangle. The

1. Required the number of shot in a complete triangular pile, one of whose sides contains 22 balls?

$$\begin{array}{r l}
22 & = \text{the number in one side of base.} \\
23 & = \text{the number} + 1. \\
\hline
66 & \\
44 & \\
\hline
506 & \\
24 & = \text{the number} + 2. \\
\hline
2024 & \\
1012 & \\
\hline
6)12144 & \\
\hline
2024 & = \text{the number of shot in the pile.}
\end{array}$$

2. Required the number of shot in a complete triangular pile, one side of whose base contains 15 balls?

Ans. 680 balls.

3. Required the number of balls in a triangular pile, each side of the base containing 30 balls? *Ans.* 4960.

PROBLEM II.

To find the number of balls in a square pile.

RULE. Multiply continually together, the number in one

---

pile has only one ball on the top, this ball rests upon 3 balls, which form the second row; and these 3 balls rest upon 6 balls, which form the third row; and these 6 balls rest upon 10 balls, and so on. Then the sum of all these balls is equal to $1 + 3 + 6 + 10 +$ &c. to n terms, n being the number of courses. But this series is $= 1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) +$ &c. ..... to n terms. The nth terms of this series is $= \frac{n.(n+1)}{2}$, and the last term but one is $= \frac{n.\overline{n-1}}{2}$, &c., and the sum of all this expression is $= \frac{n\cdot(n+1)\cdot(n+2)}{6}$, which is the rule.

side of the bottom course, that number increased by 1, and double the same number increased by 1; then one-sixth of the last product will be the answer.*

1. How many balls are in a square pile of 30 rows?

$$\begin{array}{rl} 30 & = \text{number in one side.} \\ 31 & = \text{number in one side} + 1. \\ \hline 930 & \\ 61 & = \text{twice the number in one side} + 1. \\ \hline 6)56730 & \\ \hline 9455 & \textit{Ans.} \end{array}$$

2. Required the number of shot in a complete square pile, one side of whose base contains 19? *Ans.* 2470.

3. How many shot in a finished square pile, when a side of the base contains 21 shot? *Ans.* 3311.

PROBLEM III.

To find the number of shot in a finished rectangular pile.

RULE. Add one to three times the number of shot contained in the length of the base, subtract the number of shot in the breadth of the base, multiply the remainder by the said number increased by one, and this result again by the number in the breadth; then one sixth of the last result will give the number of shot in the rectangular pile.†

---

* The top of the pile is one ball which rests upon 4 balls; and these 4 rest upon 9 balls; and these rest upon 16, and so on; all the courses then form a progression, such as $1^2 + 2^2 + 3^2 + 4^2 + 5^2 +$ &c. .... n^2. This progression is formed of the squares of the natural numbers 1, 2, 3, 4, 5, &c. ... n terms, the sum of which is $= \frac{n\cdot(+1)\cdot(2n+1)}{6}$ as shown in most books of Algebra, which affords the rule.

† The uppermost row consists of one row of balls; this row is supported by a double row, the length of which is one ball more than are contained in the uppermost row, and the breadth 1 ball more than the breadth of the uppermost row. The third rectangular course *is* 2 *balls* in length and 2 in breadth, more than the highest row con-

1. Required the number of shot in a finished rectangular pile, the length of the base containing 59, and its breadth containing 20 balls?

59 = the number of shot in the length.
3
———
177; then 177 + 1 = 178, and 178 − 20 = 158.
158 × 21 = 3318, and 3318 × 20 = 66360. Hence
66360 ÷ 6 = 11060, the *Ans.*

2. How many balls are in a rectangular complete pile, the length of the bottom course being 46, and its breadth 15?

Ans. 4960.

PROBLEM IV.

To determine the number of balls contained in a pile which is not finished, the highest course being complete, and the number of balls in each side thereof being given.

Rule. Find the number of shot which would be contained in the pile if it were complete. Find also the number

---

tains, and so on; therefore if $r =$ the number of balls in the highest row; $2(r+1) =$ the number of balls in the second course; $3(r+2)$ = the number in the third row, &c. Hence the number of balls in the whole pile is $r + 2(r+1) + 3(r+2) + \&c. + n(r+n-1)$, where $n =$ the number of balls in the breadth of the bottom course. But the expression

$$r + 2(r+1) + 3(r+2) + \&c. \ldots . n(r+n-1) =$$
$$r + 2r + 3r + \&c. \ldots . nr$$
$$+ 2 + 6 + 12 + \&c.\, n(n-1) = \frac{n\cdot(n+1)}{2}\cdot r + \frac{n\cdot(n-1)\cdot(n+1)}{3},$$

Now, put $m =$ the number of balls in the length of the base; then $m - n = r - 1$, or $r = m - n + 1$; therefore substituting $m - n + 1$ for r, we get the number of balls $= \frac{n\cdot(n+1)}{2}\cdot(m-n+1) +$
$\frac{n\cdot(n+1)\cdot(n-1)}{3} = n\cdot(u+1\left\{\frac{m-n+1}{2} + \frac{n-1}{3}\right\} =$
$\frac{n\cdot(n+1)\cdot(3m-n+1)}{6}$, which is the algebraic expression for the rule.

in that complete pile, each side of whose base contains one shot fewer than the corresponding side of the uppermost course of the unfinished pile, and the difference between these results will give the number of balls in the unfinished pile.*

1. How many shot are there in an unfinished triangular pile, a side of whose base contains 23, and a side of the uppermost course 7 shot?

$$
\begin{array}{rl}
23 & = \text{number of balls in the base.} \\
24 & = \text{number of balls in the base} + 1. \\
\hline
552 & \\
25 & \\
\hline
6)13800 & \\
\hline
2300 & = \text{number of the pile when complete.} \\
6 & \\
7 & \\
\hline
42 & \\
8 & \\
\hline
6)336 & \\
\hline
56 & \text{number of balls in the imaginary pile.}
\end{array}
$$

Therefore, 2300 — 56 = 2244 the *Ans.*

2. How many balls in an incomplete square pile, the side of the base being 24, and of the top 8? *Ans.* 4760.

3. How many balls are there in the incomplete rectangular pile of 12 courses, the length and breadth of the base being 40 and 20? *Ans.* 6146.

DETERMINING DISTANCES BY SOUND.

The velocity of sound, or the space through which it is propagated in a given time, has been very differently

* This is too obvious to require a demonstration.

estimated by philosophers who have written on this subject. We shall, however, take it to be 1142 feet in a second.

From repeated experiments it has been ascertained that sound moves uniformly, or to speak more philosophically, that the pulses of air which excite it move uniformly. The velocity of sound is the same with that of the aerial waves, and does not vary much, whether it go with the wind or against it. By the wind, no doubt, a certain quantity of air is carried from one place to another, and the sound is somewhat accelerated while its waves move through that part of the air, if their direction be the same as that of the wind. But as the velocity of sound is vastly swifter than the wind, the acceleration it will thereby receive is but inconsiderable, being at most but $\frac{1}{20}$ of the whole velocity.

The chief effect perceptible from the wind is, that it increases and diminishes the space through which sound is propagated. The utmost distance at which sound has been heard is about 200 miles. It is said that the unassisted human voice has been heard from Old to New Gibraltar, a distance of about 12 miles. Dr. Derham, placing cannon at different distances, and causing them to be fired off, observed the intervals between the flash and report, by means of which he found the velocity of sound to be as above stated.

1. Having observed the flash of a cannon, I noticed by my watch, that 5 seconds elapsed previous to my hearing the report; determine my distance from the gun?

$$\begin{array}{r} 1142 \\ 5 \\ \hline 5710 \end{array}$$ feet, *Ans.*

2. Being at sea, I saw the flash of a cannon, and counted 8 seconds between the flash and report; required the distance? *Ans.* $1\frac{7}{10}$ mile

GAUGING.

SECTION XI.

The business of gauging is generally performed by means of two instruments, namely, the gauging or sliding rule, and the guaging or diagonal rod.

1. Of the Gauging Rule.—Leadbetter's.

By this instrument is computed the contents of casks, &c. after the dimensions have been taken. It is a square rule, having various logarithmic lines on its four faces, and three sliding pieces, capable of being moved through grooves in which they fit, in three of these faces.

On the first face are delineated three lines, namely, two marked A, B, on which multiplication and division are performed; and the third marked M D, signifies malt depth, and serves to gauge malt. The middle one B is on the slider, and is a kind of double line, being marked at both edges of the slider, for applying it to both the lines A and M D. These three lines are all of the same radius, or distance from 1 to 10, each containing twice the length of the radius. A and B are numbered and placed exactly alike, each commencing at 1, which may be either 1, or 10, 100, &c. or ·1, or ·01, or ·001, &c. Whatever the 1 at the beginning is estimated at, the middle division 10 will be 10 times as much, and the last division 100 times as much. But 1 on the line M D is opposite 2220, or more exactly 2218·2 on the other lines, which number 2218·2 denotes the cubic inch in an imperial malt bushel; and its divisions numbered retrograde to those of A and B. On these two lines are also several other marks and letters: thus on the

line A are M B, or sometines only B for malt bushel, at the number 2218·2, and A for ale, at 282, the cubic inches in an old ale gallon; and on the line B, is W, for wine, at 231, the cubic inches in an old wine gallon.

These marks are now usually omitted upon the rule, since the late new act of parliament for weights and measures, and G for gallon is put at 277·274 the inches in an imperial gallon, whether of ale, wine, or spirits.

On many sliding rules are alo found *s i*, for square inscribed at ·707 the side of a square inscribed in a circle, whose diameter is 1; *s e*, for square equal at ·886, the side of a square which is equal to the same circle; and *c*, for circumference, at 3·1416, the circumference of the same circle.

On the second face, or that opposite the first, are a slider and four lines, marked D, C, D, E, at one end, and root square, root cube at the other end; the lines C and D containing respectively the squares and cubes of the opposite numbers on the lines D, D; the radius of D being double to that of A, B, C, and triple to that of E; therefore whatever the first 1 on D denotes, the first on C is its square, and the first on E its cube; that is, if D begin with 1, C and E will begin with 1; but if D begin with 10, C will begin with 100, and E with 1000; and so on.

On the line C are marked *o c* at ·0796, for the area of the circle whose circumference is 1; and *o d*, at ·7854, for the area of the circle whose diameter is 1.

On the line D, are marked G S, for gallon square at 16·65, and G R for gallon round at 18·789; also M S for malt square at 47·097, and M R for malt round at 53·144.

These are the respective gauge-points for gallons and bushels. The first 16·65 is the side of a square, which at an inch depth, holds a gallon; the second 18·789, the diameter of a circle, which at an inch depth, holds a gallon; the third 47·097 the side of a square, which at an inch depth holds a bushel; the fourth, 53·144, the diameter of a circle, which at an inch depth, holds a bushel.

On the third face are three lines, one on a slider marked N; and two on the stock, marked S S and S L, for segment standing and segment lying, which serve ullaging, standing and lying casks.

And on the fourth side, or opposite face, are a scale of inches, and three other scales, marked spheroid, or 1st variety, 2d variety, 3d variety; the scale for the fourth or conic variety, being on the inside of the slider in the third face. The use of these lines is to find the mean diameter of casks. On the insides of the two first sliders, besides all those already described, are two other lines, being continued from one slider to the other.

The one of these is a scale of inches, from $2\frac{1}{2}$ to 36, and the other is a scale of ale gallons, between the corresponding number 435 and 3·61; which form a table, to show, in ale gallons, the contents of all cylinders whose diameters are from $12\frac{1}{2}$ to 36 inches, their common altitude being 1 inch.

Veree's Sliding Rule.

This rule is in the form of a parallelopipedon, and is generally made of box.

1. The line marked A, on the face of this rule, is called Gunter's line, and is numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. At 2150·42 is fixed a brass pin, marked M B, signifying cubic inches in a bushel of malt; at 282 is fixed another brass pin, marked A′, denoting the number of cubic inches in an ale gallon.

2. The line marked B is on the other slide, and is divided exactly like that marked A. There is another slide B, which is used along with this; the two brass ends are then placed together, and so make a double radius, numbered from the left-hand towards the right. At 231, on the second radius, is a fixed brass pin, marked W, denoting the cubic inches in a gallon of wine; at 314 is fixed another brass pin, marked C, signifying the circumference of a circle whose diameter is 1. These lines are used and read exactly as the lines A and B, on the Carpenters' Rule, which have been already described.

3. The back of the first slide, or radius, marked B, has the dimensions for ale, wine, wash-tun gallons, malt, green starch, dry starch, hard soap hot, hard soap cold, green soft soap, white soft soap, flint glass, &c. &c. as in the following *table*, Problem I.

The back of the second slide, or radius, marked B, contains the gauge-points corresponding to these divisors, where S denotes squares, and C circles.

4. The line M D on the rule, denoting malt depth, is a line of numbers commencing at 2150·42, and is numbered from the left to the right-hand 2, 10, 9, 8, 7, 6, 5, 4, 3. This rule is used in malt gauging.

5. The two slides B, just described, are always used together, either with the line A, M D, or the line D, which is on the opposite face of the rule to that already described. The line D is numbered from the left-hand towards the right, 1, 2, 3, 31, 32, which is at the right-hand end; it is then continued from the left-hand end of the other edge of the rule 32, 4, 5, 6, 7, 8, 9, 10. At 17·15 is fixed a brass pin, marked W′ G, denoting the circular gauge-point for wine gallons. At 18·95 is a brass pin marked A′ G for ale gallons. At 46·37, M′ S signifies the square gauge-point for malt bushels. At 52·32, M′ B denotes the round or circular gauge-point for malt bushels. The line D on this rule is of the same nature as the line marked D on the Carpenters' Rule, which has been already described. The line A and the two slides B, are used together, which perform multiplication, division, simple proportion, &c.; and the line D, and the same slides B, are used together for extracting the square and cube roots.

6. The other two slides belonging to this rule are marked C, and are divided in the same manner, and used together, like the slides B.

The back of the first slide, or radius, marked C, is divided, next the edge, into inches, and numbered from the left-hand towards the right 1, 2, 3, 4, 5, &c., and these inches are again sub-divided into ten equal parts. The second line is marked spheroid, and is numbered from the left-hand towards the right 1, 2, 3, 4, 5, 6, 7. The third line is marked second variety, and is numbered 1, 2, 3, 4, 5, 6. These lines are used, with the scale of inches, for finding a mean diameter.

The back of the second slide, or radius, marked C, has several factors for reducing goods of one denomination to others of equivalent values. Thus | X. to VI. 6. | signifies

that to reduce strong beer at 8*s*. per barrel, to small beer at 1*s*. 4*d*., you are to multiply by 6. | VI. to X. 17. | signifies that to reduce small beer at 1*s*. 4*d*. per barrel to strong beer at 8*s*. per barrel, you are to multiply by ·17. | C 4 ꝏ to X. 27. | signifies that 27 is the multiplier for reducing cider at 4*s*. per barrel to another at 8*s*., &c.

7. The two slides C, just described, are always used together, with the lines on the rule marked Seg. St. or S S, segments standing; and Seg. L *y* or S L, segments lying; for ullaging casks. The former of these lines is numbered 1, 2, 3, 4, 5, 6, 7, 8, which stands at the right-hand end; it then goes on from the left-hand on the other edge 8, 9, 10, &c. to 100, the latter is numbered in the same manner 1, 2, 3, 4, which stands at the right-hand end; it then goes on from the left-hand on the other edge, 4, 5, 6, 7, &c. to 100.

PROBLEM I.

To find the several multipliers, divisors, and gauge-points, belonging to the several measures now used.

As 282 solid inches are contained in one gallon of ale or beer; 231 solid inches in one gallon of wine; 268·8 solid inches in a gallon of malt, or 2150·42 solid inches in a bushel of malt, of corn measure; then it is obvious that if 1 be divided by 282, 231, 268·8, and 2150·42, respectively, the quotients will be the multipliers for ale, wine, and malt gallons, and the last for malt bushels respectively.

Hence the method of finding the following multipliers is obvious :—

282)1·00000(·003546 multiplier for ale gallons.
231)1·00000(·004329 multiplier for wine gallons.
268·8)1·00000(·0037202 multiplier for malt gallons.
2150·42)1·00000(·00046502 multiplier for malt bushels.
227)1·00000(·00405 multiplier for mash-tun gallons.

Now it is manifest that if the solid inches contained in any vessel be multiplied by these multipliers, the product will be the gallons in the respective measures; or dividing by the

divisors 282, 231, 268·8, the quotient will be the gallons likewise.

It has been shown in Mensuration, that when the diameter of a circle is 1, the area of that circle is ·785398, &c., which being nearly equal to ·7854, is seldom employed, and ·7854 substituted for convenience; then by dividing the solid capacity of any figure by ·7854, the quotients will be the proper divisors to the square diameters of circular figures. Then to reduce the area at one inch deep into gallons, divide ·7854, or ·785398, &c. by 282, 231, 268·8, and the quotients will give the multipliers for ale, wine, &c., gallons respectively; and ·7854 divided into 282, 231, 2150·42, will give the divisors for the ale and wine gallons, and for the corn bushel.

The square root of the divisor is the guage-point.

282)·785398(·002785 multiplier for ale gallons.
231)·785398(·00339 multiplier for wine gallons.
·785398)282(359·05 divisor for ale gallons.
·785398)231(294·12 divisor for wine gallons.
·785398)2150·42(2738 divisor for corn bushels.

The gauge-point for squares in	Ale measure, is.....	16·79
	Wine measure, is ...	15·19
	Malt bushel, is	46·37
The gauge-point for circular figures in	Ale measure, is.....	18·95
	Wine measure, is ...	17·15
	Malt bushel, is	52·32

Any divisor for a circular figure being multiplied by 1·5, gives the divisor for spheres.

In this manner the numbers in the following Table were calculated.

A TABLE

Of Multipliers, Divisors, and Guage-points, for Squares and Circles.

Note. The Areas, &c. are all in inches.	Multipliers for		Divisors for		Gauge-points for	
	Squares.	Circles,	Squares.	Circles.	Squares.	Circles.
The side or diameter 1	1·	·785398	1·	1·27324	1·	1·128
A superficial foot	·006944	·005454	144·	183·34	12·	13·54
A solid foot	·000578	·000454	1728·	2200·16	41·57	46·91
Ale gallon	·003546	·002785	282·	359·05	16·79	18·95
Wine gallon	·004329	·003399	231·	294·12	15·19	17·15
Malt, or corn bushel	·000465	·000365	2150·42	2738·	46·37	52·32
Malt gallon	·003720	·002922	268·8	342·24	16·39	18·5
Mash-tun gallon	·004405	·00346	227·	289·	15·1	17·07
A pound of hard soap	·036845	·028939	27·14	34·56	5·21	5·88
A pound of hot soap	·035714	·028050	28·	35·65	5·29	5·97
A pound of green soap	·038956	·0306	25·67	82·68	5·06	5·72
A pound of white soft soap	·039123	·030731	25·56	32·54	5·05	5·7
A pound of tallow net	·031844	·025101	31·4	39·98	5·6	6·32
A pound of green starch	·028736	·022565	34·8	44·32	5·9	6·66
A pound of dry starch	·024813	·019491	40·3	51·3	6·35	7·16
A pound of flint glass	·094697	·074405	10·56	13·44	3·25	3·69
A pound of white glass	·071125	·05586	14·06	17·9	3·74	4·32
A pound of green glass	·082102	·064516	12·18	15·5	3·48	3·94

Note. It very often happens in the practice of gauging, that when the one given number is set to the gauge point on the sliding rule, the other given number will fall off the rule; hence in many cases it will be necessary to find a second, or new gauge-point. The second gauge-points are the square roots of 10 times the divisors in the above table. Thus for squares, the new gauge-point for ale is 53·10, for wine 48·06, for malt bushels 14·66; and for circles, the new gauge point for ale is 59·92, for wine 54·22, and for malt bushels 16·54.

PROBLEM II.

To find the area, in ale gallons, of any rectilineal plane figure.

RULE. By the rules given in Mensuration of Superficies, find the area of the figure in inches, which being divided by the proper divisor, or multiplied by the proper multiplier, will give the area in gallons.*

1. Suppose a back or cooler in the form of a parallelogram, to be 100 inches in length, and 40 in breadth; required the area in wine gallons or ale gallons?

$100 \times 40 = 4000$ the area inches, which divided by 282, and the quotient 14·184, will give the ale gallons; or if we multiply 4000 by ·003546, the product 14·184 is the number of ale gallons as before. And if 4000 be divided by 231, the quotient 17·316 is the gallons in wine measure, or if we multiply 4000 by ·004329, the product 17·316 will give the gallons in wine measure as before.

BY THE SLIDING RULE.

	On A		On B		On A		
As	282	:	40	::	100	:	14·2 nearly.
As	231	:	40	::	100	:	17·3 nearly.

2. If the side of a square be 40 inches, what is the area in wine gallons? *Ans.* 5·67 gallons.

3. If the side of a rhombus be 40 inches, and its perpendicular breadth 37 inches; required its area in wine gallons? *Ans.* 5·25.

4. What is the area of a square cooler, in gallons, ale measure, the side being 144 inches? *Ans.* 73·5392.

5. Allowing the side of a hexagon to be 64 inches, and the perpendicular from the centre to the middle of one of the

---

* The areas of plane figures, in gauging, are expressed in gallons, For there will be as many solid inches in any vessel of one inch deep. as there are superficial inches in its base. What is called in gauging a surface or area, is in reality a solid of one inch deep, which multiplied by the height, will give the whole content in gallons.

sides 55·42 inches; required its area in ale and wine gallons, and malt bushels?

Ans. { 37·73 ale gallons. 46·06 wine gallons. 4·94 malt bushels. }

PROBLEM III.

The diameter of a circular vessel being given in inches, to find its area in ale, or wine gallons, &c.

Rule. Multiply the square of the diameter by ·002785 for ale, or by ·003399 for wine; or divide the square of the diameter by 359·05 for ale, or by 294·12 for wine, the products or quotients, will give the ale or wine gallons respectively.

When it is required to find the area in any other denomination than ale or wine gallons, use the proper multiplier or divisor for the required denomination, as found in the table.

1. The diameter of a circular vessel is 32·6 inches; required the area in ale gallons?

$(32·6)^2 = 1062·76$. Then

$1062·76 \times ·002785 = 2·9598$ ale gallons; and

$1062·76 \times ·0034 = 3·6133$ wine gallons.

BY THE SLIDING RULE.

As 18·95 and 17·15 are the circular gauge-points for ale and wine gallons, say,

	On D		On B		On D		On B
As	18·95	:	1	::	32·6	:	2·96
As	17·15	:	1	::	32·6	:	3·61

2. If the diameter of a circular vessel be 10 inches, what is the area in ale and wine gallons?

Ans. ·2785 ale gallon, and ·34 wine gallon.

3. Suppose the diameter of a circular vessel is 30 inches, what is its area in ale and wine gallons?

Ans. 1·114 ale gallon, and 1·36 wine gallon.

4. What is the area in ale gallons in a round vessel whose diameter is 24 inches? *Ans.* 1·60416.

PROBLEM IV.

Given the transverse and conjugate diameter of an elliptical vessel, to find its area in ale or wine measure.

RULE. Multiply the product of the two diameters by ·002785 for ale, and by ·0034 for wine; or divide the product of the two diameters by 359·05 for ale, or by 294·12 for wine; the products, or quotients, will give the respective ale and wine gallons required.

When any other denomination is required, the proper multiplier, or divisor, in the table, is to be employed.

1. Suppose the longer diameter of an elliptical vessel is 10, and the shorter diameter 6, required the area in ale and wine gallons?

Here $10 \times 6 = 60$.
Then $60 \times \cdot002785 = \cdot1671$ ale gallon.
And $60 \times \cdot0034 = \cdot205$ wine gallon.

2. The transverse, or longer diameter of an elliptical vessel is 20; and the conjugate, or shorter diameter 10 inches; what is the area in ale and wine measure?

Ans. ·557 ale gallon, and ·68 wine gallon.

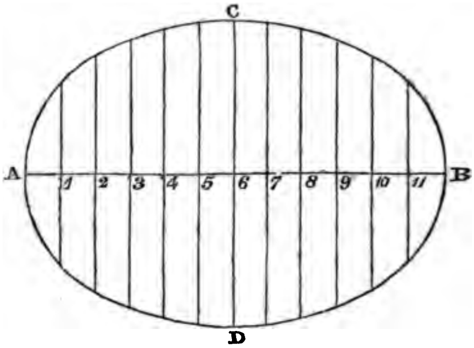
	On A		On B		On B		
As	359	:	20	::	10	:	·557 ale gallon.
As	294	:	20	::	10	:	·68 wine gallon.

3. Suppose the transverse diameter of an elliptical vessel is 70 inches, conjugate 50 inches; required its area in ale and wine gallons, and malt bushels?

Ans. $\begin{cases} 9\cdot74 \text{ ale gallons.} \\ 11\cdot9 \text{ wine gallons.} \\ 1\cdot27 \text{ malt bushel.} \end{cases}$

Note. As vessels are seldom or never made truly elliptical, being generally ovals, the area found by the above rule is not correct, except the vessel be a truly mathematical ellipsis; when the vessel is of an oval form, the area is best found by the method of equi-distant ordinates.

Let A B C D be the oval vessel whose area is required, and let A B and C D be the transverse and conjugate diameters, at right angles to each other, the former being 102·8



inches. Divide this transverse (102·8) by some even number which will leave a small remainder, and the quotient will be the distance each ordinate is to be taken from the conjugate diameter; then with chalk and a parallel ruler, draw the ordinates through the points 1, 2, 3, 4, &c. Then the area may be found by Problem XXI., Mensuration, which being multiplied or divided by the proper tabular numbers, will give the area in gallons, &c. Or,

1st. Add together the first and last ordinates.

2d. Add together the even ordinates, that is, the 2, 4, 6, 8, 10, &c., and multiply the sum by 4.

3d. Add together the odd ordinates, except the first and last; that is, the 3, 5, 7, 9, &c., and multiply the sum by 2.

4th. Multiply the sum of the extreme ordinâtes by their distance from the curve.

5th. Add the three first found sums together; and multiply the sum by the common distance of the ordinates, and to the product add the fourth found sum, and divide the total by 3, and the quotient resulting by 282·231, or 2150·42 for the area in ale and wine gallons, and malt bushels, respectively.

First, $102{\cdot}8 \div 10 = 10$ the distance of the ordinates asunder, and the remainder 2·8 is double the distance of the extreme ordinates from the curve, that is, $1{\cdot}4 = A\ 1$, or B 11.

Now, let us suppose the lengths of the ordinates to be,

1	2	3	4	5	6	7	8	9	10	11
20,	40·2,	57,	66·6,	73,	75,	73,	66·6,	57,	40·2,	20.

Then,

$$\text{1st}\begin{cases}1 = 20\\ 11 = 20\end{cases}$$

40 inches, sum of the first and last.
1·4

56

$$\text{2d}\begin{cases}2 = 40\cdot2\\ 4 = 66\cdot6\\ 6 = 75\cdot0\\ 8 = 66\cdot6\\ 10 = 40\cdot2\end{cases}$$

$$288\cdot6 \times 4 = 1154\cdot4$$

$$\text{3d}\begin{cases}3 = 57\\ 5 = 73\\ 7 = 73\\ 9 = 57\end{cases}$$

$$260 \times 2 = 520$$

Then $1154\cdot4 + 520 = 1674\cdot4$ sum of first three sums.

$$\begin{array}{r} 1674\cdot4 \\ 10 \\ \hline 16744 \\ 56 \\ \hline 3)16800 \\ \hline 5600 \end{array}$$

5600; then

$5600 \div 282 = 19\cdot85$ ale gallons.
$5600 \div 231 = 24\cdot24$ wine gallons.
$5600 \div 2150\cdot42 = 2\cdot604$ malt bushels.

When the vessel is not circular, or elliptical, it is best to measure the equi-distant ordinate, which though ever so unequal, will, by proceeding as above, serve to find the area

of the base. Whenever the vessel is an irregular curved figure, the area should be invariably found by the method of equi-distant ordinates, as the true result cannot be found by any other method.

4. What is the area, in ale and wine measure, of an ellipse, whose transverse axis is 24 and conjugate 18?

Ans. $\begin{cases} 1{\cdot}20312 \text{ ale gallon.} \\ 1{\cdot}4688 \text{ wine gallon.} \end{cases}$

PROBLEM V.

To find the content of a prism in ale or wine gallons.

RULE. Find the area of the base, by Problem II., in Gauging, which being multiplied by the depth within, will give the content in gallons.

Or, find the solid content by Mensuration, and divide that content by 282 for ale, or by 231 for wine gallons.

1. A vessel, whose base is a right-angled parallelogram, is 49·3 inches in length, the breadth 36·5 inches, and the depth 42·6 inches; required its content in ale and wine gallons?

Here $49{\cdot}3 \times 36{\cdot}5 \times 42{\cdot}6 = 76656{\cdot}57$.

Then $76656{\cdot}57 \div 282 = 271{\cdot}83$ ale gallons.

And $76656{\cdot}57 \div 231 = 331{\cdot}84$ wine gallons.

Lastly $76656{\cdot}57 \div 2150{\cdot}42 = 35{\cdot}65$ malt bushels.

BY THE SLIDING RULE.

On B		On D		On B		
49·3	:	49·3	::	36·5	:	42·42.

$$\left.\begin{matrix} 16{\cdot}79 \\ 15{\cdot}19 \\ 46{\cdot}37 \end{matrix}\right\} : 42{\cdot}6 :: 42{\cdot}42 : \begin{cases} 271{\cdot}83 \text{ ale gallons.} \\ 331{\cdot}84 \text{ wine gallons.} \\ 35{\cdot}65 \text{ malt bushels.} \end{cases}$$

2. Each side of the square base of a vessel is 20 inches, and its depth 10 inches, what is the content in ale gallons.

Ans. 14·28 gallons.

3. The side of a vessel in the form of a rhombus is 20 inches, breadth 15 inches, and depth 10 inches; required the content in ale gallons. *Ans.* 10·638 gallons.

4. What is the content of a vessel in the form of a rhomboid, whose longest side is 20 inches, breadth from side to side 8 inches, and depth 10 inches?

Ans. 6·88 wine gallons, and 5·67 ale gallons.

PROBLEM VI.

To find the content of any vessel, whose ends are squares, or rectangles, of any dimensions.

Rule. Multiply the sum of the lengths of the two ends by the sum of their breadths, to which add the areas of the two ends; this sum multiplied by $\frac{1}{6}$ of the depth, will give the solidity in cubic inches; then divide by 282, 231, or 2150·42 for the content in ale and wine gallons, and malt bushels.

1. Suppose the top and bottom of a vessel are parallelograms, the length of the top is 40 inches, and its breadth 30 inches; the length of the bottom is 30 inches, and its breadth 20; and the depth 60 inches; required the contents in ale and wine gallons?

$40 + 30 = 70$ sum of the lengths.
$30 + 20 = 50$ sum of the breadths.

3500 product.

$40 \times 30 = 1200$ area of the greater base.
$30 \times 20 = 600$ area of the less base.

5300
10 one-sixth of the depth.

53000 solidity in cubic inches.

Then $53000 \div 282 = 187·9432$.
And $53000 \div 231 = 229·4372$.

BY THE SLIDING RULE.

Find a mean proportional $\sqrt{(40 \times 30)} = 34·64$), between the length and breadth at the top, and a mean proportional $\sqrt{(30 \times 20)} = 24·49$), between the length and

breadth at the bottom; the sum of these is 59·13, twice a mean proportional between the length and breadth in the middle. Then,

$$\begin{array}{llll} \text{On D} & \text{On B} & \text{On D} & \text{On B} \\ 16{\cdot}79 : & \frac{60}{6} :: & \left\{\begin{array}{l} 34{\cdot}64 : \\ 24{\cdot}49 : \\ 59{\cdot}13 : \end{array}\right. & \left.\begin{array}{l} \text{———} \\ \text{———} \\ \text{———} \end{array}\right\} 187{\cdot}94 \text{ ale gallons.} \end{array}$$

By using the wine gauge-point, the content in wine gallons may be found.

2. Suppose the top and bottom of a vessel are parallelograms, the length of the top is 100 inches, and its breadth 70 inches; the length of the bottom 80, and its breadth 56, and the depth 42 inches; what is its content in ale and wine gallons?

Ans. { 847·94 ale gallons.
1035·15 wine gallons.

THE GAUGING, OR DIAGONAL ROD.

The diagonal rod is a square rule, having four faces, and is generally 4 feet long. It folds together by joints. This instrument is employed both for gauging and measuring casks, and computing their contents; and that from one dimension only, namely, the diagonal of the cask, or the length from the middle of the bung-hole to the meeting of the cask with the stave opposite the bung; being the longest line that can be drawn from the middle of the bung-hole to any part within the cask.

On one face of the rule is a scale of inches for measuring this diagonal; to which are placed the areas, in ale gallons, of circles to the corresponding diameters, in like manner as the lines on the under sides of the three slides in the sliding rule.

On the opposite face, there are two scales of ale and wine gallons, expressing the contents of casks having the corresponding diagonals.

All the other lines on this instrument are similar to those on the sliding rule, and are used in the same manner.

Example. The diagonal or distance between the middle of the bung-hole to the most distant part of the cask, as found by the diagonal rod, is 34·4 inches; what is the content in gallons?

To 34·4 inches correspond, on the rod, 90¾ ale gallons, or 111 wine gallons, 92½ imperial gallons, the content required.

Note. The contents shown by the rod answer to the most common form of casks, and fall in between the 2nd and 3rd varieties following.

Of Casks as divided into Varieties.

Casks are usually divided into four varieties, which are easily distinguished by the curvature of their sides.

1. The middle frustrum of a spheroid belongs to the first variety.

2. The middle frustrum of a parabolic spindle belongs to the second variety.

3. The two equal frustrums of a paraboloid belong to the third variety.

4. And the two equal frustrums of a cone belong to the fourth variety.

If the content of any of these be found in inches by their proper rules, and this divided by 277·271, or 2218·2, the quotient will be the content in imperial gallons, or bushels, respectively.

PROBLEM VII.

To find the content of a vessel in the form of the frustrum of a cone.

Rule. To three times the product of the two diameters, add the square of their difference; multiply the sum by ⅓ of the depth, and divide the product by 359·05 for ale, and by 294·12 for wine, and by 2738 for malt bushels.

1. What is the content of a cone's frustrum, whose greater diameter is 20 inches, least diameter 15 inches, and depth 21 inches?

$20 \times 15 \times 3 = 900$
$20 - 15 = 5$ & $5^2 = 25$

$925 \times 7 = 6475$. Then

$359{\cdot}05)6475(18{\cdot}03$ ale gallons.
$294{\cdot}12)6475(22{\cdot}01$ wine gallons.

2. The greater diameter of a conical frustrum is 38 inches, the less diameter 20·2, and depth 21 inches; what is the content in ale gallons? *Ans.* 51·07 gallons.

PROBLEM VIII.

To find the content of the frustrum of a square pyramid.

RULE. To three times the product of the top and bottom sides, add the square of their difference, multiply their sum by $\frac{1}{3}$ of the depth, and divide the product by 282 and 231, for ale and wine gallons, respectively.

1. Suppose the greater base is 20 inches, the less base 15 inches, and depth 21 inches; required the content in wine measure?

$20 \times 15 \times 3 = 900$
$20 - 15 = 5$
Then $5 \times 5 = 25$

$925 \times 7 \div 231 = 27{\cdot}8$ gallons.

Note. The contents of the frustrum of a pyramid is found just like that of a cone, with the exception of the tabular divisor, or multiplier, the cone requiring the circular factor, and the pyramid the square one.

PROBLEM IX.

To find the content of a globe.

RULE. Multiply the diameter of the globe by its circumference, and the resulting product by $\frac{1}{6}$ of the diameter; then the last product multiplied or divided by the circular factor, will give the contents in gallons.

1. Let the diameter be 34 inches, what is its contents?

$34 \times 34 \times 34 \times \cdot5236 = 20579\cdot5744$.

Then $20579\cdot5744 \div 282 = 72\cdot9772$ ale gallons.

And $20579\cdot5744 \div 231 = 89\cdot08$ wine gallons.

RULE II. Or cube the diameter of the globe, which multiply by ·001856 (⅔ of ·002785) for the contents in ale gallons, and by ·002266 for wine gallons.

$34^3 = 39304$; then $39304 \times \cdot001856 = 72\cdot948$ ale gall.
$34^3 = 39304$: then $39304 \times \cdot002266 = 89\cdot062$ wine gallons.

2. What are the contents of a globe in ale and wine measure, the diameter being 20 inches?

Ans. { 14·848 ale gallons. 18·128 wine gallons.

3. Required the contents of a globular vessel, whose diameter is 100 inches?

Ans. { 1856 ale gallons. 2266 wine gallons.

PROBLEM X.

To find the contents of the segment of a sphere, as the rising crown of a copper still, &c.

RULE. Measure the diameter, or chord of the segment, and the altitude just in the middle. Multiply the square of half the diameter by 3; to the product add the square of the altitude; multiply this sum by the altitude, and the product again by ·00856, or ·002266, for ale and wine measure, respectively.

Here $27\cdot6 \div 2 = 13\cdot8$.

Then $13\cdot8 \times 13\cdot8 \times 3 = 571\cdot32$
$9\cdot2 \times 9\cdot2 = 84\cdot64$

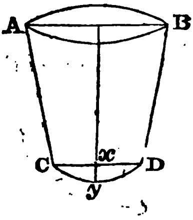
655·96 sum.
9·2 depth.

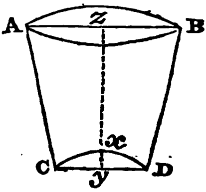
$6034\cdot832 \times \cdot001856 = 11\cdot2006$ ale gallons, and $6034\cdot832 \times \cdot002266 = 13\cdot675$ wine gallons.

PROBLEM XI.

To gauge a copper, having either a concave or convex bottom; or what is called a falling bottom, or rising crown.

RULE. If the side of the vessel be straight with a falling bottom, find the content of the segment C y D, by Prob. X.; find also the content of the upper part A B D C, by Prob. VII.; the sum of both will give the contents of the copper.





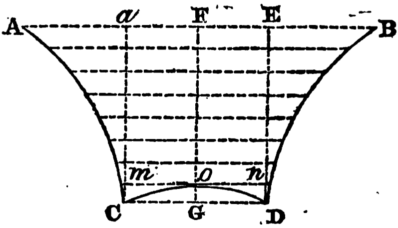
When the copper has a rising crown, find the contents of A B C D, by Prob. VII., from which deduct the contents of the segment C x D, and the remainder will be the contents of the vessel A B D x C.

PROBLEM XII.

To guage a vessel whose side is curved from top to bottom.

Take the diameters at equal distances of 2, 3, 4, or 5 inches, according as the case may require; that is, if the side of the vessel be considerably curved, the greater the number of diameters that will be required; the less the curvature of the side, the less the number of diameters that will be required; in this case, the diameters being taken at the distance of 6 inches, the result will be sufficiently accurate for all practical purposes.

To gauge the vessel, or copper, A B D C, fasten a piece of pack-thread at A and B, as A F B; then with some con-



venient instrument find the distance a C of the deepest part of the copper, which let us suppose to be 47 inches.

By means of the same instrument measure the distance o F from the top of the crown to F the middle of A B; which let us suppose to be 42 inches, this deducted from a C, 47, will leave 5· (= o G) the height of the crown.

To find the diameter C D, *of the bottom of the crown.*

Measure the top diameter A B, which suppose to be 99 inches; then hold a thread, so that a plummet attached to the end thereof, may hang just over C, and measure A a = B E, each of which let us admit to be 17·5 inches; add these together, and deduct their sum (35) from 99, and the remainder (64) will evidently be equal to C D, the diameter at the bottom of the crown. Measure the diameter $m\,o\,n$, which touches the top of the crown, which suppose is 65 inches.

Now as this copper is not considerably curved, the diameters may be taken in the middle of every 6 inches of the depth, which suppose to be as in the second column of the following table; to each diameter find the area in gallons, by Prob. III., which write in the third column; find also the content of every 6 inches, corresponding to these diameters, which write in the fourth column of the table; lastly, find the contents of the crown by Prob. X., and subtract it from the contents of A B D G C, the remainder will give the capacity of the copper.

Or find the contents of A B n m, and C D being 64 inches, the area answering to which is 11·408, this multiplied by half the altitude of the crown, viz. by 2·5, gives 28·52 gallons, the content of the crown. The contents of the part

m n D C is 57·935 gallons, from which the contents of the crown being deducted, the remainder (29·415 gallons) is the quantity of liquor which covers the crown.

TABLE IX.

Parts of the depth	Diameters.	Areas.	Content of every 6 inches
6	95·3	25·2945	151·767
6	90·1	22·6095	135·657
6	85·0	20·1223	120·734
6	80·0	17·8246	106·947
6	75·2	15·7499	94·499
6	70·5	13·8426	83·056
6	66·0	12·1310	72·791
	The sum............		765·451
	To cover crown		29·415
	The whole content		794·866

PROBLEM XIII.

To find the content of any close cask.

Whatever be the form of the cask, the following dimensions must be taken; that is,

$$\left.\begin{array}{l}\text{The bung diameter,}\\ \text{The head diameter,}\\ \text{The length of the cask,}\end{array}\right\}\ \text{within.}$$

On account of the difficulty in ascertaining the figure of the cask, it is not, in many cases, easy to find the exact contents of casks.

In taking the dimensions of a cask, it is essential that the bung-hole be in the middle of the cask, and also, that the bung-stave, and the stave opposite to it, are both regular and even within.

It is likewise essential that the heads of casks are equal and truly circular; and if so, the distance between the inside of the chimb to the outside of its opposite stave, will be the head diameter within the cask, nearly.

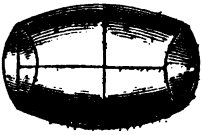
From the variety in the forms of casks, no general rule

could be given to answer every form; two casks may have equal head diameters, equal bung diameters, and equal lengths, and yet their contents may be very unequal.

PROBLEM XIV.

To find the content of a cask of the first form.

RULE. To the square of the head diameter, add double the square of the bung diameter; and multiply the sum by the length of the cask. Then multiply the last product by $\cdot0009\frac{4}{9}$, or divide by 1059·1, the product or quotient will be the content in imperial gallons.



1. What is the content of a spheroidal cask, whose length is 40 inches, bung diameter 32 inches, and head diameter 24 inches?

$$
\begin{array}{r}
24 \times 24 = 576 \\
32 \times 32 = 1024 \\
2 \\
\hline
2048 \\
576 \\
\hline
2624 \times 40 = 104960 \\
\cdot0009\frac{4}{9} \\
\hline
944640 \\
34987 \\
11662 \\
\hline
99\cdot1289 \text{ imperial gallons.}
\end{array}
$$

BY THE GAUGING RULE.

Set 40 on C, to the G R 18·79 on D, against
24 on D, stands 64·99 on C,
32 on D, stands 116·2 on C,
$+$ 116·2

$$
\begin{array}{r}
3)297\cdot39 \\
\hline
99\cdot13 \text{ gallons.}
\end{array}
$$

2. What is the content of a spheroidal cask, whose length is 20 inches, bung diameter 16 inches, and head diameter 12 inches?

Ans. { 12·36 ale gallons.
14·869 wine gallons.

To find the content of a cask by the mean diameter.

RULE. Multiply the difference of the head and bung diameters by ·68 for the first variety; by ·62 for the second variety; by ·55 for the third; and by ·5 for the fourth, when the difference between the head and bung diameters is less than 6 inches; but when the difference between these exceeds 6 inches, multiply that difference by ·7 for the first variety; by ·64 for the second; by ·57 for the third; and by ·52 for the fourth. Add this product to the head diameter, and the sum will be a mean diameter. Square this mean diameter, and multiply the square by the length of the cask; this product multiplied, or divided, by the proper multiplier, or divisor, will give the content.

By resuming the second-last example, we have

Bung diameter	32	29·6	mean diameter.
Head diameter	24	29·6	
	8	876·16	square.
	·7	40	length.
	5·6	359·5)35046·40	
	24		
		97·6	gallons.
mean diameter	29·6		

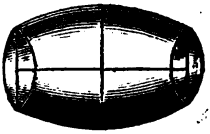
In the same manner the content for the second variety will be 94·46 ale gallons; for the third variety 90·87 ale gallons; and for the fourth variety 83·34 gallons.

PROBLEM XV.

To find the content of a cask of the second form.

RULE. To the square of the head diameter, add double the square of the bung diameter, and from the sum deduct

$\frac{2}{5}$ of the square of the difference of the diameters; multiply the remainder by the length, and the product again by $\cdot 0009\frac{4}{9}$ for the content in imperial gallons.



1. What is the content of a cask, whose length is 40 inches, bung diameter 32 inches, and head diameter 24 inches?

$$32 - 24 = 8;\ \text{then}\ 8^2 = 64,\ \text{and}\ \tfrac{2}{5}\ \text{of}\ 64 = 25{\cdot}6$$
$$24^2 = 576,\ \text{and}\ 32^2 = 1024,\ \text{then}\ 1024 \times 2 = 2048$$
$$2048 + 576 = 2624,\ \text{and}\ 2624 - 25{\cdot}6 = 2598{\cdot}4$$
$$40$$
$$\overline{103936,}$$

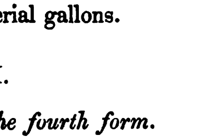
$103936 \times \cdot 0009\frac{4}{9} = 98{\cdot}1617$ gallons.

PROBLEM XVI.

To find the contents of a cask of the third form.

Rule. To the square of the bung diameter, add the square of the head diameter; multiply the sum by the length, and the last product by $\cdot 001416$ for the answer in imperial gallons.

Let us resume the last example:

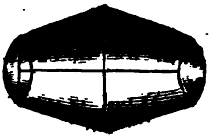


$$32^2 = 1024$$
$$24^2 = 576$$
$$\overline{1600} \times 40 = 64000$$
$$\cdot 001416$$
$$\overline{90{\cdot}624}\ \text{imperial gallons.}$$

PROBLEM XVII.

To find the content of a cask of the fourth form.

Rule. Add the square of the difference of the diameters, to 3 times the square of their sum; multiply the sum by the length, and the last product by $\cdot 000236$ for the content in gallons.



Resuming still the last example, 32 + 24 = 56, and $56^2 \times 3 = 3136 \times 3 = 9408$, and $8^2 = 64$, then 9408 + 64 = 9472; then $9472 \times 40 = 378880$, and $378880 \times \cdot000236 =$ 89·41668 imperial gallons.

PROBLEM XVIII.

To find the content of any cask by Doctor Hutton's general rule.

RULE. Add into one sum, 39 times the square of the bung diameter, 25 times the square of the head diameter, and 26 times the product of the two diameters; then multiply the sum by the length, and the product again by $\cdot00031\frac{4}{9}$ for the content in gallons.

1. What is the content of a cask, whose length is 40 inches, and the bung and head diameters 32 and 24?

$$32^2 = 1024 \qquad 24^2 = 576 \qquad 32 \times 24 = 768$$
$$\begin{array}{r} 1024 \\ 39 \\ \hline 39936 \end{array} \qquad \begin{array}{r} 576 \\ 25 \\ \hline 14400 \end{array} \qquad \begin{array}{r} 768 \\ 26 \\ \hline 19968 \end{array}$$

$$\begin{array}{r} 39936 \\ 14400 \\ 19968 \\ \hline 74304 \end{array}$$

$$74304 \times 40 = 2972160$$
$$\begin{array}{r} 2972160 \\ \cdot00031\frac{4}{9} \\ \hline 93\cdot4579 \text{ gallons.} \end{array}$$

ULLAGING.

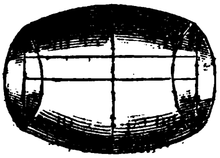
PROBLEM XIX.

To ullage a lying cask.

This is the finding what quantity of liquor is contained in a cask when partly empty.

To ullage a cask, the wet and dry inches must be known, as also the content of the cask and bung diameter.

RULE. Take the wet inches, and divide them by the bung diameter; find the quotient in the column of versed sines, in the Table at the end of the book, taking out its corresponding segment; multiply this segment by the whole content of the cask, and the product arising by $1\frac{1}{4}$ for the ullage required, nearly.



1. Find the ullage for 8 wet inches, the bung diameter being 32 inches, and the content 92 ale gallons?

32)8(.25 whose tabular segment is ·153546.

Then $\cdot153546 \times 92 = 14\cdot126232$.

And $14\cdot126232 \times 1\frac{1}{4} = 17\cdot65779$ gallons.

PROBLEM XX.

To ullage a standing cask.

RULE. Add together the square of the diameter at the surface of the liquor, the square of the diameter of the nearest end, and the square of double the diameter taken in the middle between the other two; multiply the sum by the length between the surface and nearest end, and the product arising by ·000472 for the gallons in the less part of the cask whether empty or filled.

1. What is the ullage for 10 wet inches, the three diameters being 24, 27, and 29 inches?

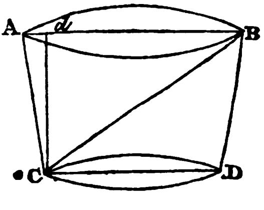
$$\begin{array}{rr}
24^2 = & 576 \\
29^2 = & 841 \\
(2 \times 27)^2 = & 2916 \\
\hline
 & 4333 \\
 & 10 \\
\hline
 & 43330
\end{array}
\qquad
\begin{array}{r}
43330 \\
\cdot000472 \\
\hline
86660 \\
303310 \\
173320 \\
\hline
20\cdot45176 \text{ gallons.}
\end{array}$$

PROBLEM XXI.

To find the content of an ungula, or hoof, of the frustrum of a cone.

RULE. For the less hoof, multiply the product of the less diameter and height, by the product of the greater diameter multiplied by a mean proportional between both diameters, less the square of the less diameter, and this last divided by three times the circular factor multiplied by the difference of the diamaters, gives the content of the less hoof.

1. $CD = 30$, $AB = 40$, $Cd = 20$, required the content of the less hoof?



$40 \times 30 = 1200$, and $\sqrt{1200} = 34{\cdot}6$ mean.

$30 \times 20 = 600$, 1st product.

$40 \times 34{\cdot}6 = 1384$, 2nd product.

$30 \times 30 = 900$

484 remains.

$$484 \times 600 = 290400$$

$40 - 30 = 10$, then $359 \times 3 \times 10 = 10770$, and $290400 \div 10770 = 26{\cdot}96$ gallons, for the greater hoof ABC.

RULE. Multiply the product of the greater diameter and the height of the frustrum, by the square of the greater diameter made less by the product of the less diameter multiplied by a mean proportional between those diameters; this remainder, divided by three times the circular divisor multiplied by the difference of the diameters, give the content of the greater hoof.

Resuming the last example, we have

$$40 \times 40 = 1600$$

$20 \times 40 = 800$ 1st product.

$40 \times 30 = 1200$, and $\sqrt{1200} = 34{\cdot}6$

$34{\cdot}6 \times 30 = 1038$ 2nd product.

$$40 - 30 = 10.$$

$$\text{Then } 1600 - 1038 = 562$$
$$800$$

$$359 \times 3 \times 10 = 10770)449600 \text{ last product.}$$
$$41{\cdot}74 \quad \text{gallons of ale.}$$

PROBLEM XXII.

To gauge a Still.

Fill the still with water, and draw it off into another vessel of some regular form, whose content is easily computed. This is by far the most accurate method that can be employed.

Or gauge the shoulder by itself, and gauge the body by taking a great number of diameters at near and equal distances throughout, first covering the bottom, if there be any cavity, with water, the quantity of which is known.

A TABLE

OF THE AREAS OF THE SEGMENTS OF A CIRCLE,

Whose diameter is 1, *and supposed to be divided into* 1000 *equal parts.*

Height	Area Seg.	Height	Area Seg.	Height	Area Seg.	Height	Area Seg.
·001	·000042	·038	·009763	·075	·026761	·112	·048262
·002	·000119	·039	·010148	·076	·027289	·113	·048894
·003	·000219	·040	·010537	·077	·027821	·114	·049528
·004	·000337	·041	·010931	·078	·028356	·115	·050165
·005	·000470	·042	·011330	·079	·028894	·116	·050804
·006	·000 18	·043	·011734	·080	·029435	·117	·051446
·007	·000779	·044	·012142	·081	·029979	·118	·052090
·009	·000951	·045	·012554	·082	·030526	·119	·052736
·009	·001135	·046	·012971	·083	·031076	·120	·053385
·010	·001329	·047	·013392	·084	·031629	·121	·054036
·011	·001533	·048	·013818	·085	·032180	·122	·054689
·012	·001746	·049	·014247	·086	·032745	·123	·055345
·013	·001968	·050	·014681	·087	·033307	·124	·056003
·014	·002199	·051	·015119	·088	·033872	·125	·056663
·015	·002438	·052	·015561	·089	·034441	·126	·057326
·016	·002685	·053	·016007	·090	·035011	·127	·057991
·017	·002940	·054	·016457	·091	·035585	·128	·058658
·018	·003202	·055	·016911	·092	·036162	·129	·059327
·019	·003471	·056	·017369	·093	·036741	·130	·059999
·020	·003748	·057	·017831	·094	·037323	·131	·060672
·021	·004031	·058	·018296	·095	·037909	·132	·061348
·022	·004322	·059	·018766	·096	·038496	·133	·062026
·023	·004618	·060	·019239	·097	·039087	·134	·062707
·024	·004921	·061	·019716	·098	·039680	·135	·063389
·025	·005230	·062	·020196	·099	·040276	·136	·064074
·026	·005546	·063	·020681	·100	·040875	·137	·064760
·027	·005867	·064	·021168	·101	·041476	·138	·065449
·028	·006194	·065	·021659	·102	·042080	·139	·066140
·029	·006527	·066	·022154	·103	·042687	·140	·066833
·030	·006865	·067	·022652	·104	·043296	·141	·067528
·031	·007209	·068	·023154	·105	·043908	·142	·068225
·032	·007558	·069	·023659	·106	·044522	·143	·068924
·033	·007913	·070	·024168	·107	·045139	·144	·069625
·034	·008273	·071	·024680	·108	·045759	·145	·070328
·035	·008638	·072	·025195	·109	·046381	·146	·071033
·036	·009008	·073	·025714	·110	·047005	·147	·071741
·037	·009383	·074	·026236	·111	·047632	·148	·072450

Height	Area Seg.	Height	Area Seg.	Height	Area Seg.	Height	Area Seg.
·149	·073161	·198	·110226	·247	·150953	·296	·194509
·150	·073874	·199	·111024	·248	·151816	·297	·195422
·151	·074589	·200	·111823	·249	·152680	·208	·196337
·152	·075306	·201	·112624	·250	·153546	·299	·197252
·153	·076026	·202	·113426	·251	·154412	·300	·198168
·154	·076747	·203	·114230	·252	·155280	·301	·199085
·155	·077469	·204	·115035	·253	·156149	·302	·200003
·156	·078194	·205	·115842	·254	·157019	·303	·200922
·157	.078921	·206	·116650	·255	·157890	·304	·201841
·158	·079649	·207	·117460	·256	·158762	·305	·202761
·159	·080380	·208	·118271	·257	·159636	·306	·203683
·160	·081112	·209	·119083	·258	·160510	·307	·204605
·161	·081846	·210	·119897	·259	·161386	·308	·205527
·162	·082582	·211	·120712	·260	·162263	·309	·206451
·163	·083320	·212	·121529	·261	·163140	·310	·207376
·164	·084059	·213	·122347	·262	·164019	·311	·208301
·165	·084801	·214	·123167	·263	·164899	·312	·209227
·166	·085544	·215	·123988	·264	·165780	·313	·210154
·167	·086289	·216	·124810	·265	·166663	·314	·211082
·168	·087036	·217	·125634	·266	·167546	·315	·212011
·169	·087785	·218	·126459	·267	·168430	·316	·212940
·170	·088535	·219	·127285	·268	·169315	·317	·213871
·171	·089287	·220	·128113	·269	·170202	·318	·214802
·172	·090041	·221	·128942	·270	·171089	·319	·215733
·173	·090797	·222	·129773	·271	·171978	·320	·216666
·174	·091554	·223	·130605	·272	·172867	·321	·217599
·175	·092313	·224	·131438	·273	·173758	·322	·218533
·176	·093074	·225	·132272	·274	·174649	·323	·219468
·177	·093836	·326	·133108	·275	·175542	·324	·220404
·178	·094601	·227	·133945	·276	·176435	·325	·221340
·179	·095366	·228	·134784	·277	·177330	·326	·222277
·180	·096134	·229	·135624	·278	·178225	·327	·223215
·181	·096903	·230	·136465	·279	·179122	·328	·224154
·182	·097674	·231	·137307	·280	·180019	·329	·225093
·183	·098447	·232	·138150	·281	·180918	·330	·226033
·184	·099221	·233	·138995	·282	·181817	·331	·226974
·185	·099997	·234	·139841	·283	·182718	·332	·227915
·186	·100774	·235	·140688	·284	·183619	·333	·228858
·187	·101553	·236	·141537	·285	·184521	·334	·229801
·188	·102334	·237	·142387	·286	·185425	·335	·230745
·189	·103116	·238	·143238	·287	·186329	·336	·231689
·190	·103900	·239	·144091	·288	·187234	·337	·232634
·191	·104685	·240	·144944	·289	·188140	·338	·233580
·192	·105472	·241	·145799	·290	·189047	·339	·234526
·193	·106261	·242	·146655	·291	·189955	·340	·235473
·194	·107051	·243	·147512	·292	·190864	·341	·236421
·195	·107842	·244	·148371	·293	·191775	·342	·237369
·196	·108636	·245	·149230	·294	·192684	·343	·238318
·197	·109430	·246	·150091	·295	·193596	·344	·239268

Height	Area Seg.	Height	Area Seg.	Height	Area Seg.	Heigh	Area Seg.
·345	·240218	·384	·277748	·423	·316004	·462	·354736
·346	·241169	·385	·278721	·424	·316992	·463	·355732
·347	·242121	·386	·279694	·425	·317981	·464	·356730
·348	·243074	·387	·280668	·426	·318970	·465	·357727
·349	·244026	·388	·281642	·427	·319959	·466	·358725
·350	·244980	·389	·282617	·428	·320948	·467	·359723
·351	·245934	·390	·283592	·429	·321938	·468	·360721
·352	·246889	·391	·284568	·430	·322928	·469	·361719
·353	·247845	·392	·285544	·431	·323918	·470	·362717
·354	·248801	·393	·286521	·432	·324909	·471	·363715
·355	·249757	·394	·287498	·433	·325900	·472	·364713
·356	·250715	·395	·288476	·434	·326892	·473	·365712
·357	·251673	·396	·289453	·435	·327882	·474	·366710
·358	·252631	·397	·290432	·436	·328874	·475	·367709
·359	·253590	·398	·291411	·437	·329866	·476	·368708
·360	·254550	·399	·292390	·438	·330856	·477	·369707
·361	·255510	·400	·293369	·439	·331850	·478	·370706
·362	·256471	·401	·294349	·440	·332843	·479	·371705
·363	·257433	·402	·295330	·441	·333836	·480	·372704
·364	·258395	·403	·296311	·442	·334829	·481	·373703
·365	·259357	·404	·297292	·443	·335822	·482	·374702
·366	·260320	·405	·298273	·444	·336816	·483	·375702
·367	·261284	·406	·299255	·445	·337810	·484	·376702
·368	·262248	·407	·300238	·446	·338804	·485	·377701
·369	·263213	·408	·301220	·447	·339798	·486	·378701
·370	·264178	·409	·302203	·448	·340793	·487	·379700
·371	·265144	·410	·303187	·449	·341787	·488	·380700
·372	·266111	·411	·304171	·450	·342782	·489	·381699
·373	·267078	·412	·305155	·451	·343777	·490	·382699
·374	·268045	·413	·306140	·452	·344772	·491	·383699
·375	·269013	·414	·307125	·453	·345768	·492	·384699
·376	·269982	·415	·308110	·454	·346764	·493	·385699
·377	·270951	·416	·309095	·455	·347759	·494	·386699
·378	·271920	·417	·310081	·456	·348755	·495	·387699
·379	·272890	·418	·311068	·457	·349752	·496	·388699
·380	·273861	·419	·312054	·458	·350748	·497	·389699
·381	·274832	·420	·313041	·459	·351745	·498	·390699
·382	·275803	·421	·314029	·460	·352742	·499	·391699
·383	·276775	·422	·315016	·461	·353739	·500	·392699

LAND-SURVEYING.

SECTION XII.

Land-surveying is that art which enables us to give a true plan or representation of any field or parcel of land, and to determine the superficial contents thereof.

In measuring land, the area or superficial content is always expressed in acres, or in acres, roods and perches; each acre containing 4 roods, and each rood 40 perches.

Land is measured with a chain, called Gunter's chain, of 4 poles or 22 yards in length, which consists of 100 equal links, each link being $\frac{22}{100}$ of a yard long, or $\frac{66}{100}$ of a foot, or 7·92 inches. 10 square chains, or 10 chains in length and 1 in breadth, make an acre; or 4840 square yards, 160 square poles, or 100,000 square links make an acre. The length of lines measured with a chain, are generally set down in links as integers; every chain being 100 links in length. Therefore, after the content is found, it will be in square links, and as 100,000 square links make an acre, it will be necessary to cut off five of the figures on the right-hand for decimals, and the rest will be acres. The decimals are reduced to roods by multiplying by 4, and cutting off five figures as before for decimals, which decimal part is reduced to perches by multiplying by 40, and cutting off five figures from the product. As an example:

Suppose the length of a rectangular piece of ground to be 792 links, and its breadth 385; required the number of acres, roods, and perches it contains?

792	3·04920
385	4
3960	·19680
6336	40
2376	7·87200
304920	

A. R. P.
Ans. 3 : 0 : 7

The statute perch is $5\frac{1}{2}$ yards, but the Irish plantation perch is 7 yards; hence the length of a plantation chain is 10·08 inches.

PROBLEM I.

To measure a line, or distance on the ground, two persons are employed, the foremost, for the sake of distinction, is called the leader, and the hindermost, the follower.

Ten small arrows, or rods, to stick in the ground at the end of each chain, are provided; also some station-staves, or long poles with coloured flags, to set up in the direction of the line to be measured, if there do not appear some marks naturally in that direction.

The leader takes the 10 arrows in one hand, and one end of the chain, by the ring, in the other; the follower stands at the beginning of the line, holding the ring at the end of the chain in his hand, while the leader drags forward the chain by the other end of it, till it is stretched straight, and the leader directed, by the follower, by moving his hand, to the right or left, till the follower see him exactly in a line with the mark or direction to be measured to; then both of them holding the chain level and stretched, the leader sticks an arrow upright in the ground, as a mark for the follower to come to, and advances another chain forward, being directed in his position by the follower standing at the arrow, as before, as also by himself, now and at every succeeding chain's length, by moving himself from side to side, till the follower and back mark be in a direct line. Having then stretched the chain, and stuck down an arrow,

as before, the follower takes up his arrow, and thus they proceed till tne 10 arrows are employed, or in the hands of the follower, and the leader, without an arrow, is arrived at the end of the eleventh chain-length. The follower then sends or brings the 10 arrows to the leader, who puts one of them down at the end of his chain, and advances with his chain, as before. And thus the arrows are changed from one to the other at every 10 chains length, till the whole line is finished, if it exceed 10 chains; and the number of changes shows how many times ten chains the line contains, to which the follower adds the arrows he holds in his hand, and the number of links of another chain over to the mark or end of the line. Thus, if the whole line measure 36 chains 45 links, or 3645 links, the arrows have been changed three times, the follower will have 5 arrows in his hand, the leader 4, and it will be 45 links from the last arrow, to be taken up by the follower, to the end of the line.

In works on Surveying, it is usual to describe the various instruments used in the art. The pupil, however, will best learn the use of these instruments when actually engaged in the practice. The chief instruments employed are the chain, the plain table, the theodolite, the cross, the circumferentor, the off-set staff, the perambulator, used in measuring roads, and other great distances.

Levels, with telescopic or other sights, are used to find the level between two or more places, or how much one place is higher or lower than another.

Besides all these, various scales are used in protracting and measuring on paper; such as plane scales, line of chords, protractor, compasses, reducing scales, parallel and perpendicular rulers, &c.

The Field Book.

In surveying with the plain table, a field-book is not required, as every thing is drawn on the table immediately when it is measured. But when the theodolite, or any other instrument is used, some sort of a field-book is used in order to register all that is done relative to the survey in hand. This book every one contrives and rules as he thinks fit. It is, however, usually divided into three columns. The middle

column contains the different distances on the chain-line, angles, bearings, &c., and the columns on the right and left are for the off-sets on the right and left, which are set against their corresponding distances in the middle column; as also for such remarks as may occur, and may be proper to note in drawing the plan; such as houses, ponds, castles, churches, rivers, trees, &c. &c.

But in smaller surveys, an excellent way of setting down the work, is to draw by the eye, on a piece of paper, a figure resembling that which is to be measured; and then write the dimensions, as they are found, against the corresponding parts of the figure. This method may be practised, even in larger surveys, and is far superior to any other at present practised. A specimen of this plan will be seen further on.

FORM OF THE FIELD BOOK.

Offsets and remarks on the left.	Stations, Bearings, and Distances.	Offsets and remarks on the right.
	☐ 1	
	104° 25′	
	00	
Cross a hedge 24	67	Brown's barn.
a brook 30	120	
	734	
	954	Tree.
	736	67 stile.
	82	
	62° 25′	
	00	
House corner 61	40	
	67	
Foot-path 15	84	44
	95	
	467	14 Spring.
	976	
	☐ 3	
	54° 17′	
	62	20 Pond.
	124	
Clayton's hedge 24	630	
	767	
	767	30 Stile.
	305	
	760	

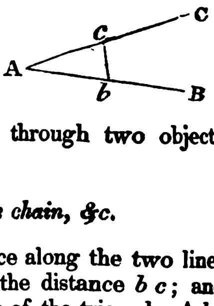
In this form of a field-book □ 1 is the first station, where the angle or bearing is 104° 25′. On the left at 67 links in the distance or principal line, is an offset of 24; and at 120 an offset of 30 to a brook on the right; at 67 Brown's barn is situated; at 954 is an offset of 20 to a tree, and at 736 an offset to a stile.

And so on for the other stations.

A line is drawn under the work, at the end of every station, to prevent confusion.

PROBLEM II.

To take angles and bearings.



Let it be required to take the bearings of the two objects B, C, from the station A.

In this Problem it is required to measure the angle at A, formed by two lines, passing from the station A, through two objects B and C.

1. *By measurement with the chain, &c.*

Measure, with the chain, any distance along the two lines A B, A C, as A *b*, A *c*; then measure the distance *b c*; and this being done, transfer the three sides of the triangle A *b c* to paper, on which measure the angle *c* A *b*, as in Problem XV. Practical Geometry.

2. *With the magnetic needle and compass.*

Turn the instrument, or compass, so that the north end of the needle may point to the flower-de-luce. Then direct the sights to a mark at B, noting the degrees cut by the needle. Next direct the sights to another mark at C, noting the degrees cut by the needle, as before. Then their sum or difference, as the case may be, will give the number of degrees in the angle C A B.

3. *With the theodolite, &c.*

Direct the fixed sights along the line A B, by turning the instrument about till you see the mark B through these sights, and then screw the instrument fast. Then turn the moveable index about till, through its sights, you see the other mark C. Then the degrees cut by the index, on the graduated limb or ring of the instrument, show the number of degrees in the angle C A B.

4. *With the plain table.*

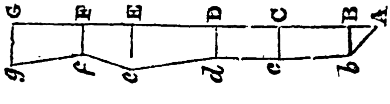
Having covered the table with paper, and fixed it on its stand, plant it at the station A, and fix a fine pin, or a point of the compass in a proper point of the paper, to represent the station A. Close by the side of this pin, lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights; then by the fiducial edge of the index draw a line. By a similar process draw another line in the direction of the object C. And it is done.

PROBLEM III.

To measure the offsets.

Let A *b c d e f g* be a crooked hedge, river, or brook, &c., and A G a base line.

Begin at the point A, and measure towards G; and when you come opposite any of the corners *b c d*, &c. which is



ascertained by means of the cross-staff, measure the offsets B *b*, C *c*, D *d*, &c. with the chain, and register the dimension, as in the annexed field-book.

FIELD-BOOK.

Off-sets Left.	Base line A G, or ☐, Station.	Off-sets Right.
91	785 = A G.	——
57	634	——
98	510	——
70	340	——
84	220	——
62	45	——
—	☐ A go North.	——

To lay down the plan.

Draw the line A G of an indefinite length; then by a diagonal scale, set off A B equal to 45 links; at B erect the perpendicular B *b* equal to 62 links taken from the same scale. Next set off A C equal to 220 links, or 2 chains 20 links, and at C erect the perpendicular C *c*, equal to 84 links, in the same way set off A C equal to 340 links, or 3 chains 40 links, and at D erect the perpendicular D *d* equal to 70 links. Proceed in a similar manner with the remaining offsets, and a line joining A *b c d e*, &c. will complete the figure.

To find the content.

Some authors direct to add up all the perpendiculars B *b*, C *c*, &c. and divide their sum by the number of them, then multiply the quotient by the length A G. This method, however, should never be used, except when the offsets B *b*, C *c*, &c. are equally distant from each other.

When the offsets are not equally distant from each other, which indeed is generally the case, this method is erroneous; therefore the following method ought to be employed.

Find the content of the space A B C as a triangle, by Problem V., Section II. Find the contents of the figures B C *c b*, C D *d c*, &c. as trapezoids, by Problem XIII., Section II., the sum of all these separate results will be the content of the figure A G *g f e d c b* A.

The actual calculation is as follows:

CALCULATION.

AB = 45	AC = 220	AD = 340	AE = 510	AF = 634	AG = 784
B*b* = 62	AB = 45	AC = 220	AD = 340	AE = 510	AF = 634
90	BC = 175	CD = 120	DE = 170	EF = 124	GF = 151
270					
	B*b* = 62	C*c* = 84	D*d* = 70	E*e* = 98	F*f* = 57
2790	C*c* = 84	D*d* = 70	E*e* = 98	F*f* = 57	G*g* = 91
	Sum 146	Sum 154	Sum 168	Sum 155	Sum 148
	BC = 175	CD = 120	DE = 170	EF = 124	FG = 151
	Prod. 25550	18480	28560	19220	22348

These respective products are evidently double the true contents of the respective figures AB*b*, BC*cb*, CD*dc*, &c. that is,

2790 = double area of AB*b*.
25550 = double area of BC*cb*.
18480 = double area of CD*dc*.
28560 = double area of DE*ed*.
19220 = double area of EF*fe*.
22348 = double area of FG*gf*.

2)116948 = double area of the whole in square links.

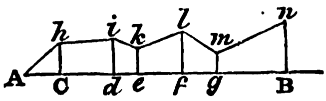
58474 = area in square links.

·58474 = area in acres = 0A. 2R. 13·5584P.

2. Required the plan and content of part of a field, from the following field-book:

A C 45	62 *c h*
A *d* 220	84 *d i*
A *e* 340	70 *e k*
A *f* 510	88 *f l*
A *g* 634	57 *g m*
A B 785	91 B *n*

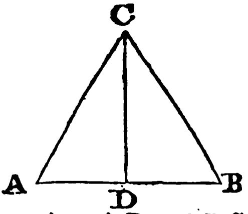
Ans. 0A. 2R. 12P.



PROBLEM IV.

To measure a field of a triangular form.

1. *By the chain.*



Set up marks at the three corners A, B, C, and measure, with the chain, the distance A D, D being the point at which a perpendicular demitted from C, would meet the line A B; measure also the distance D B; hence you have the measure of A B. Next measure the perpendicular D C; then from the two dimensions A B and D C, the content may be found by Problem IV. Section II.

Let A D $=$ 794, A B $=$ 1321, B C $=$ 826 links.

$1321 \times 826 \div 2 = 545573$ links.

Then $545573 \div 100000 = 5{\cdot}45573$ acres.

$45573 \times 4 = 1{\cdot}82292$ rood.

$82292 \times 40 = 32{\cdot}91680$ perches.

Hence the answer is 5 A. 1 R. 33 P. nearly.

2. What is the area of a triangular field, whose base is 12·25 chains, and height 8·5 chains? *Ans.* 5 A. 0 R. 33 P.

2. *By taking one or more of the angles.*

Measure two sides A B, A C, and the angle A, included between them; then find the area by note at page 34.

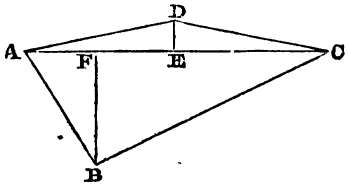
Or, measure the two angles A and B, and the adjacent side A B, from which the figure may be planned, and the perpendicular C D found, which perpendicular being multiplied by half the base A B, will give the area. Or by measuring the three sides of the triangle, its area may be found by Problem V. Section II.

PROBLEM V.

1. *By the chain.*

To survey a four-sided field.

Measure the diagonal A C, and, as before directed, measure the perpendiculars D E and B F; then the area of each of the triangles A B C, A D C may be found, as in the last



Problem, and both areas being added together, will give the content of the four-sided figure A B C D.

1. Let A C = 592, D E = 210, B F = 306 links.

$$592 \times 210 = 124320 \text{ double area of A D C.}$$
$$592 \times 306 = 181152 \text{ double area of A B C.}$$

$$2)305472 \text{ double area of A B C D.}$$

$$1{\cdot}52736 = \text{area of A B C D.}$$
$$\phantom{1{\cdot}5273}4$$

$$2{\cdot}10944$$
$$\phantom{2{\cdot}109}40$$

$$4{\cdot}37760$$

Hence 1 A. 2 R. 4 P. the answer.

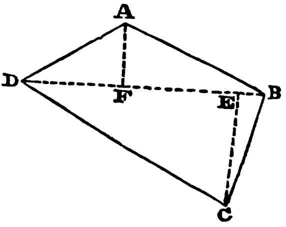
2. *By taking one or more of the angles.*

Measure the diagonal A C, also the sides A D and A B. Next measure the angles D A C and B A C; then the area of each of the triangles A B C and A D C may be found by note at page 34.

2. Required the plan and content of a field by the following field-book:

FIELD-BOOK.

——	1360 = A B.	——
——	1190	625
342	600	——
——	□ D go East.	——
Off-sets Left.	Station □, or Base line.	Off-sets Right.



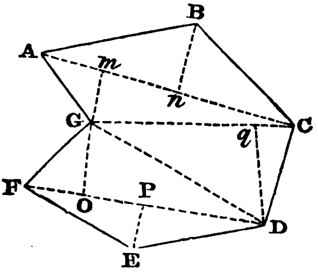
Ans. 6 A. 2 R. 12 P.

3. How many acres are there in a four-sided field, whose diagonal is 4·75 chains, and the two perpendiculars falling on it, from its opposite angles, 2·25 and 3·6 chains, respectively? *Ans.* 13 A. 3 R. 23 P.

PROBLEM V.

To survey a field of many sides by the chain only.

Let A B C D E F G be the field whose content is required. Set up marks at the corners of the field, if there be none there naturally. Consider how the field may be best divided



into trapeziums and triangles; measure them separately, as in the two last Problems; and the sum of all the separate results will give the area of the whole field.

In this way of measuring with the chain, the field should be divided into trapeziums and triangles, by drawing diagonals from corner to corner, so as that all the perpendiculars may be within the figures.

The last figure is divided into two trspeziums A B C G, G D E F, and the triangle G C D. In the first trapezium measure the diagonal A C, and the two perpendiculas G *m* and B *n*. In the triangle G C D, measure the base G C, and the perpendicular D *q*. Finally, measure the diagonal F D, and the two perpendiculars G *o* and E *p*. Having drawn a rough figure resembling the field, set all these measures against the corresponding parts of the figure. Or set them down thus:

A *m*	135	130 *m* G
A *n*	415	180 *n* B
A C	550	
C *q*	152	230 *q* D
C G	440	
F *o*	206	120 *o* G
F *p*	288	80 *p* E
F D	520	

CALCULATION.

$130+180=310$, $550\div2=275$, $275\times310=85250=$ A B C G.

$440\times230\div2=50600=$ C G D.

$120+80=200$, $520\div2=260$, $260\times200=52000=$ D E F G.

$1{\cdot}87850=$ A B C D E F G.

$$\begin{array}{r} 1{\cdot}87850 \\ 4 \\ \hline 3{\cdot}51400 \\ 40 \\ \hline 20{\cdot}56000 \end{array}$$

1 A. 3 R. 20·56 P. *Ans.*

Other methods will naturally present themselves to an ingenious practitioner who has read the preceding part of this work, or who has been previously acquainted with the principles of Mathematics. Every surveyor ought to be well acquainted with Plane Geometry at least. This, with a knowledge of Trigonometry, would be sufficient for the purpose of most surveyors.

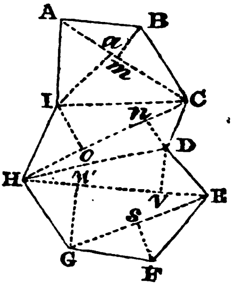
The content of the last figure may be found by measuring the sides A B, B C, C D, D E, E F, F G, G A; and the

diagonals AC, CG, GD, DF, by which the figure is divided into triangles, the content of each of which may be found by Problem V. Section II.

2. *Required the plan and content of a field of an irregular form from the following*

FIELD-BOOK.

——	900 = EG	——
268	550	——
——	□ F, go S.W.	——
——	1100 = HE	——
280	790	——
——	350	410
——	□ H, go East	——
——	1180 = GH	——
——	710	280
140	350	——
——	□ C, go S.W.	——
——	900 = AC	——
200	430	——
——	300	450
——	□ A, go S.E.	——
Off-sets Left.	Stations, □, or Base Lines.	Off-sets Right.



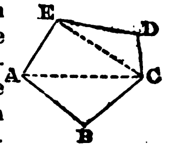
Ans. 10 A. 1 R. 24·64 P.

PROBLEM VII.

To survey field with the Theodolite, &c.

1. *From one point or station.*

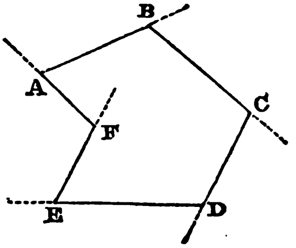
When all the angles can be seen from one point, as suppose C.



Having placed the instrument at C, turn it about till, through the fixed sights, the mark B may be seen. Fixing the instrument in this position, turn the moveable index about, till the mark A is seen through the sights, and note the degrees on the instrument. In the same manner, turn the index successively to the angles E and D, taking care to note the degrees cut off at each; by which you have all the angles, viz. B C A, B C E, B C D. Now, having obtained the angles, measure the lines C B, C A, C E, C D; entering the respective measures against the corresponding part of a rough figure, drawn to resemble the figure.

2. *By going round the field.*

Set up marks at B, C, D, &c. Place the instrument at the point A, and turn it about till the fixed index be in the direction A B, and then screw it fast: turn the moveable index in the direction A F, and the degrees cut off will be



the angle A; next measure A B, and planting the instrument at B, measure, as before, the angle B; measure the line B C, and the angle C; and so proceed round the figure, always measuring the side as you go along, as also the angles.

The 32nd Proposition of the 1st Book of Euclid, affords an easy method of proving the work: thus, add all the internal angles, A, B, C, &c. of the figure together, and their sum must be equal to twice as many right angles as the figure has sides, wanting four right angles. But when the figure

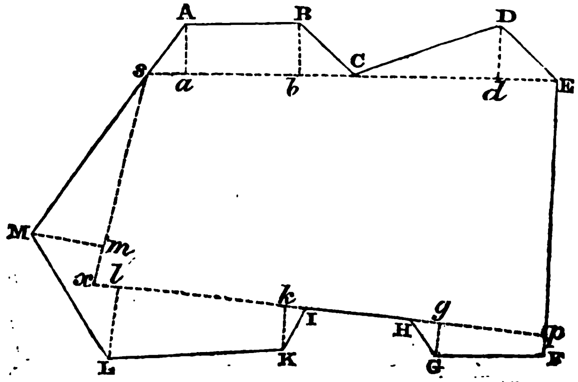
has a re-enterant angle, as F, measure the external angle, which is less than two right-angles, and deduct it from four right-angles, or 360 degrees, the remainder will give the internal angle, (if such it may be called,) which is greater than 180 degrees.

When the field is surveyed from one station, as in the first case shown above, the content of the figure is found by the note at the foot of page 34; as we have two sides and the angle included between them in each triangle of the figure.

PROBLEM VIII.

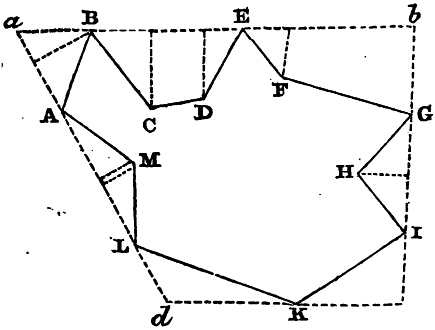
To survey a field with crooked hedges.

Measure the lengths and positions of imaginary lines, running as near the sides of the field as you can; and in proceeding along these lines, measure the off-sets to the different corners, as before taught, and join the ends of the off-sets; these connecting lines will represent the required figure. When the plane table is used, the plan will be truly represented on the paper which covers it. But when the survey is made with the theodolite, or other instrument, the different measures are to be noted in the field-book, from which the sides and angles are laid down on a map after returning from the field.



In surveying the piece A B C D E F G H I K L M, set up marks at *s* E F *x*. Begin at the station *s*, and measure the lines *s* E, E F, F *x*, *x s*, as also their positions, or the angles E *s x*, *s* E F, E F *x*, and F *x s*; and in going along the four-sided figure *s* E F *x*, measure the off-sets at *a*, *b*, *d*, *g*, *k*, *l*, *m*, as before taught. By means of the figure *s* E F *x*, and of the off-sets, the ground is easily planned.

When the principal lines are taken within the figure, as in the above case, the contents of the exterior portions *s* C B A, C D E, &c. must be added to the area of the quadralateral *s x* F E. But when the principal lines are taken outside the figure, the portions included between them and the boundaries of the field, are to be deducted from the content of the quadralateral, and the remainder will give the true content of the field.



When there are obstructions within the figure, such as wood, water, hills, &c., measure the lengths and positions of the four-sided figure *a b c d*, taking care to measure the off-sets from the different corners as you go along.

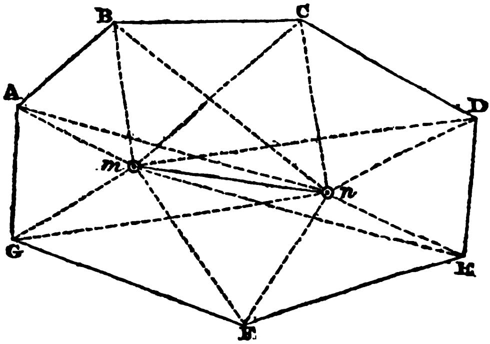
PROBLEM IX.

To survey any piece of land, by two stations.

Choose two stations, from which all the corners of the ground can be seen, if possible; measure the distance between the stations; at each station take the angles formed

by every object, from the station line or distance. Then, the station line, and these different angles being laid down, from a regular scale, and the external points of intersection connected, the connecting lines will give the boundary.

The two stations may be taken within the bounds, in one of the sides, or without the bounds of the ground to be surveyed.



Let *m* and *n* be two stations, from which all the marks A, B, C, &c. can be seen, plant the instrument at *m*, and by it, measure the angles A*m n*, B *m n*, C *m n*, &c. Next measure *m n*, and planting the instrument at *n*, measure the A *n m*, B *n m*, C *n m*, &c. These observations being planned, the lines joining the points of external intersection, will give a true map of the ground. The method of finding the content will be shown further on.

The principal objects on the ground may be delineated on the map, by measuring the angles, at each station, which every object makes with the station line *m n*. When all the objects to be surveyed cannot be seen from two stations, then three or four may be used, or as many as may be found necessary; taking care to measure the distance from one station to another; placing the instrument at every station, and observing the angles formed by all the visible objects with the respective station line; then the intersection of the lines forming these respective angles, will give the positions of all the remarkable objects thus observed.

In this manner may very extensive surveys be taken; and the positions of hills, rivers, coasts, &c., ascertained.

PROBLEM X.

To survey a large estate.

The following method of surveying a large estate was first given by Emerson, in his Surveying, page 47. It has been followed by Hutton and Keith.

When the estate is very large, and contains a great number of fields, it cannot be accurately surveyed and planned by measuring each field separately, and then adding all the separate results together; nor by taking all the angles, and measuring the boundaries that enclose it. For in these cases, the small errors will be so multiplied as to render it very much distorted.

1. Walk over the estate two or three times, in order to get a perfect idea of its figure. And to help your memory, make a rough draft of it on paper, inserting the names of the different fields within it, and noting down the principal objects.

2. Choose two or more elevated places in the estate, for your stations, from which you can see all the principal parts of it; and let these stations be as far distant from each other as possible, as the fewer stations you have to command the whole, the more exact the work will be.

In selecting the stations, care should be taken that the lines which connect them may run along the boundaries of the estate, or some of the hedges to which off-sets may be taken when necessary.

3. Take such angles, between the stations, as you think necessary, and measure the distance from station to station, always in a right line; these things must be done till you get as many lines and angles as are sufficient for determining all the station points. In measuring any of these station distances, mark accurately where these lines meet with any hedges, ditches, roads, lanes, paths, riverlets, &c., and where any remarkable object is placed, by measuring its

distance from the station line; and where a perpendicular from it cuts that line; and always mind, in any of these observations, that you be in a right line, which you may easily know by taking a back-sight and fore-sight, along the station line. In going along any main station line, take off-sets to the ends of all hedges, and to any pond, house, mill, bridge, &c., omitting nothing that is remarkable. All these things must be noted down; for these are the data by which the places of such objects are to be determined on the plan.

Be careful to set up marks at the intersections of all hedges with the station-line, that you may know where to measure from when you come to survey the particular fields that are crossed by this line.

These fields must be measured as soon as you have completed your station-line, whilst they are fresh in your memory. In this manner all the station lines must be measured, and the situations of all adjacent objects determined. It will be proper to lay down the work on paper every night, that you may see how you go on.

4. With respect to the internal parts of the estate, they must be determined by new station lines; for, after the main stations are determined, and every thing adjoining to them, then the estate must be sub-divided into two or three parts by new station lines; taking the inner stations at proper places, where you can have the best view. Measure these station lines as you did the first, and all their intersections with hedges, ditches, roads, &c., also take off-sets to the bends of hedges, and to such objects as appear near these lines. Then proceed to survey the adjoining fields, by taking the angles which the sides make with the station line at the intersections, and measuring the distances to each corner from these intersections; for, every station line will be a basis to all future operations; the situation of every object being entirely dependent on them, and therefore they should be taken of as great length as possible; and it is best for them to run along some of the hedges or boundaries of one or more fields, or to pass through some of their angles.

All things being determined for these stations, you must take more inner stations, and continue to divide and subdivide, till at last you come to single fields; repeating the

same work for the inner stations as for the outer ones, till the whole is finished. The oftener you close your work, and the fewer lines you makes use of, the less you will be liable to error.

5. An estate may be so situated that the whole cannot be surveyed together, because one part of the estate may not be seen from another. In this case you may divide it into three or four parts, and survey these parts separately, as if they were lands belonging to different persons, and at last join them together.

6. As it is necessary to protract or lay down the work as you proceed in it, you must have a scale of due length to do it by. To get such a scale, measure the whole length of the estate in chains; then consider how many inches long the map is to be; and from these you will know how many chains you must have in an inch; then make your scale accordingly, or choose one already made.

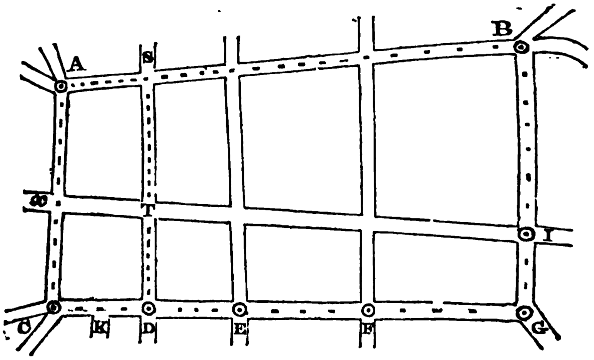
7. The trees in every hedge-row may be placed in their proper situation, which is soon done by the plane table; but may be done by the eye without an instrument; and being thus taken by guess in a rough draft, they will be exact enough, being only to look at; except it be such as are at any remarkable places, as at the ends of hedges, at stiles, gates, &c., and these must be measured or taken with the plane table, or some other instrument. But all this need not be done till the draft is finished. And observe, in all hedges, what side the gutter or ditch is on, and to whom the fence belongs.

PROBLEM XI.

To survey a town or city.

To survey a town or city, it will be proper to have an instrument for taking angles, such as a theodolite or plane table; the latter is a very convenient instrument, because the minute parts may be drawn upon it on the spot. A chain of 50 feet long, divided into 50 links, will be more convenient than the common surveying chain, and an off-set staff of 10

feet long will be very useful. Begin at the meeting of two or more of the principal streets, through which you can have the longest prospects, to get the longest station lines. There having fixed the instruments, draw lines of direction along these streets, using two men as marks, or poles set in wooden pedestals, or perhaps some remarkable places in the houses at the farther ends, as windows, doors, corners, &c. Measure these lines with the chain, taking off-sets with the staff, at all corners of streets, bendings, or windings, and to all remarkable objects, as churches, markets, halls, colleges, eminent buildings, &c. Then remove the instrument to another station, along one of these lines, and there repeat the same process as before. And so continue until the whole is finished.



Thus, fix the instrument at A, and draw lines in the directions of all the streets meeting there; then measure A C, noting the street at x. At the second station C, draw the directions of all the streets meeting there; measure from C to D, noting the place of the street K, as you pass by it. At the third station D, take the direction of all the streets meeting there, and measure D S, noting the cross street at T. Proceed in like manner through all the principal streets, after which proceed to the smaller and intermediate streets; and last of all to the lanes, alleys, courts, yards, and every other place which it may be thought proper to represent in the *plan.*

PROBLEM XII.

To compute the contents of any survey.

1. In small and separate pieces, the method generally employed is, to compute their contents from the measures of the lines taken in surveying them, without drawing any correct map of them: rules for this purpose have been given in the preceding part of the work. But in large pieces, and whole estates, consisting of a great number of fields, the usual method is, to make an unfinished but correct plan of the whole, and from this plan, the boundaries of which include the whole estate, compute the contents quite independent of the measures of the lines and angles that were taken in surveying. Divide the plan of the survey into triangles and trapeziums, by drawing new lines through it; measure all the bases and perpendiculars of all these new figures, by means of the scale from which the plan was drawn, and from these dimensions compute the contents, whether triangles, or trapeziums, by the proper rules for finding the area of such figures.

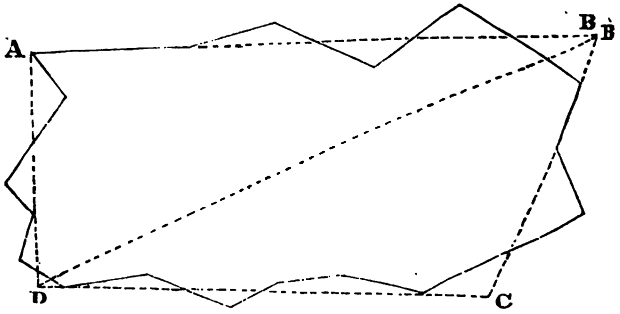
The chief difficulty in computing consists in finding the contents of land bounded by curved or very irregular lines, or in reducing such crooked sides or boundaries to straight lines, that shall enclose an equal area with those crooked sides, and so obtain the area of the curved figure by means of the right-lined one, which in general will be a trapezium.

The reduction of crooked sides to straight ones, is easily performed, thus:

Apply a horse-hair or silk thread across the crooked sides in such a manner, that the small parts cut off from the crooked figure by it, may be equal to those taken in. A little practice will enable you to exclude exactly as much as you include; then, with a pencil, draw a line along the thread, or horse-hair. Do the same by the other sides of the figure, and you will thus have the figure reduced to a straight-sided figure equal to the curved one; the content of which, being computed, as before directed, will be the content of the curved figure proposed.

The best way of using the thread or horse-hair is, to string a small slender bow with it, either of whalebone or wire, which will keep it stretched.

If it were required to find the contents of the following crooked-sided figure; draw the four dotted straight lines A B, B C, C D, and D A, excluding as much from the survey as is taken in by the straight lines; by which the crooked figure is reduced to a right-lined one, both equal in area. Then draw the diagonal B D, which being measured by a proper scale, and multiplied by half the sum of the perpendiculars let fall from A and C upon B D, (measured on the same scale) will give the area required.



Many other methods might have been given for computing the contents of a survey, but they are omitted, the above being, perhaps, themost expeditious.

MISCELLANEOUS PROBLEMS.

1. The three sides of a triangle are 12, 20, and 28; what is the area? *Ans.* $60\sqrt{3}$.

2. Find the difference between the area of a triangle whose sides are 3, 4, and 5 feet; and the area of an equilateral triangle having an equal perimeter? *Ans.* ·928 of a square foot.

3. There is a segment of a sphere, the diameter of whose base is 24 inches, and its altitude 10 inches; required its solidity? *Ans.* 2785·552 inches.

4. There is a bushel in the form of a cylinder, whose depth is 8 inches, and breadth $18\frac{1}{2}$ inches; required to determine the breadth of another cylindrical vessel of the same capacity as the former, whose depth is only $7\frac{1}{2}$ inches? *Ans.* 19·107 inches.

5. Required the length of a cord, one end of which is fastened to a stake, the other to an ass, which is allowed to feed on just an acre of grass? *Ans.* 117 feet 9 inches.

6. A ladder, 40 feet long, may be so planted, that it shall reach a window 33 feet from the ground on one side of the street; and, by only turning it over, without moving the foot out of its place, it will do the same by a window 21 feet high on the other side; what is the breadth of the street? *Ans.* 56 feet $7\frac{3}{4}$ inches.

7. In turning a one-horse chaise within a ring of a certain diameter, it was observed that the outer wheel made two turns while the inner made but one; the wheels were both

4 feet high; and supposing them fixed at the statutable distance of 5 feet asunder on the axle-tree, what was the circumference of the track described by the outer wheel?

Ans. 63 feet, nearly.

8. A cable which is 3 feet long, and 9 inches in compass, weighs 22 lb. what will a fathom of that cable weigh, which measures a foot about? *Ans.* $78\frac{2}{9}$ lb.

9. How many small cubes, a side of which equals 4 inches, may be cut out of a large cube, whose side is 8 inches? *Ans.* 8.

10. Determine the areas of an equi-lateral triangle, a square, a hexagon, a dodecagon, and a circle; the perimeter of each being 40 feet? *Ans.* 76·980035, 100, 115·47.

11. A person wants a cylindrical vessel 3 feet deep, that shall contain twice as much as another cylindrical vessel whose diameter is $3\frac{1}{2}$ feet, and altitude 5 feet; find the diameter of the required vessel? *Ans.* 6·39 feet.

12. Three persons having bought a conical sugar-loaf, wish to divide it into three equal parts by sections parallel to the base; it is required to find the altitude of each person's share, the altitude of the loaf being 20 inches?

Ans. Altitude of the upper part = 13·867, of the middle part = 3·604, of the lower part 2·528 inches.

13. There is a frustum of a pyramid, whose bases are regular octagons; each side of the greater base is 21 inches, and each side of the less base 9 inches, and its perpendicular length 15 inches, how many solid feet are contained in it?

Ans. 119·2 feet.

14. Requiring to find the height of a May-pole, I procured a staff 5 feet in length, and placing it in the sunshine, perpendicular to the horizon, I found its shadow to be 4·1 feet. Next I measured the shadow of the May-pole, which I found to be 65 feet; from this data the height of the pole is required? *Ans.* 79·26 feet.

15. Given two sides of an obtuse angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land?

Ans. 58·876 or 23·099.

16. A circular fish-pond is to be made in a garden, that shall take up just half an acre; what must be the length of the chord that strikes the circle? *Ans.* $27\frac{1}{4}$ yards.

17. A gentleman has a garden 100 feet long, and 80 feet broad; now a gravel walk is to be made of an equal width all round it; what must the breadth of the walk be, to take up just half the ground? *Ans.* 25·968 feet.

18. A silver cup, in form of the frustum of a cone, whose top diameter is 3 inches, its bottom diameter 4, and its altitude 6 inches, being filled with liquor, a person drank out of it till he could see the middle of the bottom; it is required to find how much he drank? *Ans.* ·152127 ale gallon.

19. I have a right cone, which cost me £5 13*s.* 7*d.* at 10*s.* a cubic foot, the diameter of its base being to its altitude as 5 to 8; and would have its convex surface divided in the same ratio, by a plane parallel to the base; the upper part to be the greater; required the slant height of each part?

Ans. { 3·9506486 the slant height of the upper part.
1·0854612 the slant height of the under part.

20. How many acres of the earth's surface may be seen from the top of a steeple whose height is 400 feet, the earth being supposed a perfect sphere, whose circumference is 25000 miles? *Ans.* 12120981·338267112 acres.

21. Two boys meeting at a farm-house had a tankard of milk set down to them; the one being very thirsty drank till he could see the centre of the bottom of the tankard; the other drank the rest. Now, if we suppose that the milk cost $4\frac{1}{2}d.$, and the tankard measure 4 inches diameter at the top and bottom, and 6 inches in depth; it is required to know what each boy has to pay, proportionable to the quantity of milk he drank?

Ans. { 14·1802815 farthings, for the first.
3·8197185 farthings, for the second.

22. If the linear side of a certain cube be increased one inch, the surface of the cube will be increased 246 square inches; determine the side of the cube? *Ans.* 20 inches.

23. If from a piece of tin, in the form of a sector of a circle, whose radius is 30 inches, and the length of its arc 36 inches, be cut another sector whose radius is 20 inches;

and if then the remaining frustum be rolled up so as to form the frustum of a cone; it is required to find its content, supposing one-eighth of an inch to be allowed off its slant height for the bottom, and the same allowance of the circumference, of both top and bottom, for what the sides fold over each other, in order to their being soldered together?

Ans. 685·3263 cubic inches.

24. Three men bought a grinding-stone of 40 inches diameter, which cost 20*s.* of which sum the first man paid 9*s.*, the second 6*s.*, and the third 5*s.*; how much of the stone must each man grind down, proportionably to the money he paid?

Ans. The first man must grind down 5·167603 inches of the radius; the second 4·832397 inches, and the third 10 inches.

25. There is a frustum of a cone, whose solid content is 20 feet, and its length 12 feet; the greater diameter is to the less as 5 to 2; what are the diameters?

Ans. $\begin{cases} 2\cdot02012 \text{ feet.} \\ \cdot80804 \text{ feet.} \end{cases}$

26. A farmer borrowed of his neighbour part of a hay-rick, which measured 6 feet in length, breadth, and thickness; at the next hay-time he paid back two equal cubical pieces, each side of which was 4 feet. Has the debt been discharged? *Ans.* No; 88 cubic feet are due.

27. There is a bowl in form of the segment of an oblong spheroid, whose axes are to each other in the proportion of 3 to 4, the depth of the bowl one-fourth of the whole transverse axis, and the diameter of its top 20 inches; it is required to determine what number of glasses a company of 10 persons would have in the contents of it, when filled, using a conical glass, whose depth is 2 inches, and the diameter of its top an inch and a half?

Ans. 114·0444976 glasses each.

28. If a cubical foot of brass were to be drawn into wire, of $\frac{1}{40}$ of an inch in diameter; it is required to determine the length of the said wire, allowing no loss in the metal?

Ans. $55\frac{5}{9}$ miles.

29. How many shot are there in an unfinished oblong

pile, the length and breadth of whose base being 48 and 30, and the length and breadth of the highest course being 24 and 6? *Ans.* 17356.

30. How many shot are there in an unfinished oblong pile of 12 courses; length and breadth of the top contain 40 and 10 shot respectively? *Ans.* 8606 shot.

31. Of what diameter must the bore of a cannon be cast, for a ball of 24 pounds weight, so that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball?
Ans. 5·757098 inches.

32. What is the content of a tree, whose length is $17\frac{1}{4}$ feet, and which girts in five different places as follows, viz. in the first place 9·43 feet, in the second 7·92, in the third 6·15, in the fourth 4·74, in the fifth 3·16? *Ans.* 42·5195.

33. What three numbers will express the proportions subsisting between the solidity of a sphere, that of the circumscribing cylinder, and circumscribing equi-lateral cone?
Ans. 4, 6, 9.

34. Given the side of an equi-lateral triangle 10, it is required to find the radii of its circumscribing circle?
Ans. 5·7736.

35. Given the perpendicular of a plane triangle 300, the sum of the two sides 1150, and the difference of the segments of the base 495; required the base and the sides?
Ans. 945, 375, and 780.

THE END.

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